Assortment Optimization

- (Recall) Choice Model. Given $S \subseteq N = \{1, 2, \ldots, n\}$ (products)
  - $\Pi_j(S) = \mathbb{P}[\text{Prod } j \text{ purchased from } S]$
  - MNL choice models - $\exists\, v_i > 0$ s.t

$$\Pi_j(S) = \frac{v_j \, \mathbb{1}_{\{j \in S\}}}{v_0 + v(S)}, \quad v_0 \equiv \text{'attractiveness' of no purchase}, \quad v(S) \equiv \sum_{i \in S} v_i$$

  - Mixture of MNL

$$\Pi_j(S) = \sum_{g \in G} \alpha^g \left(\frac{v_j^g}{v_0^g + v^g(S)} \right), \quad \sum_{g \in G} \alpha^g = 1$$

- The assortment optimization problem
  - Exogenous prices (profits) $P_j \ \forall\, j \in N$
  - $R^* = \max_{S \subseteq N} R(S) = \max_{S \subseteq N} \sum_{j \in S} P_j \Pi_j(S)$

* Assortment Opt under MNL model (from last lecture)

- $R(S) = \sum_{j \in S} \frac{p_j v_j}{v_0 + v(S)}$, $R^* = \max_{S \subseteq N} R(S)$

Thm - Let $p_1 \geq p_2 \geq \cdots \geq p_n$

(Nested-by-revenue Sets) $E_0 = \phi$, $E_1 = \{1\}$, $E_2 = \{1,2\}$, $\ldots$, $E_n = N$

Then $\exists\, k^* \in \{0, 1, \ldots, n\}$ s.t $E_{k^*} \in \arg\max_{S \subseteq N} R(S)$

Pf - By definition $R^* \geq \sum_{j \in S} \frac{p_j v_j}{v_0 + v(S)}$ $\forall S \subseteq N$

$$\Rightarrow v_0 R^* \geq \sum_{j \in S} v_j (p_j - R^*) \quad \forall S \subseteq N$$

$\because \exists$ some $S \subseteq N$ s.t $R(S) = R^*$

$$\Rightarrow \arg\max_{S \subseteq N} R(S) = \arg\max \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\}$$

- Thus we want to find $S \in \arg\max \left\{ \sum_{j \in S} (p_j - R^*) v_j \right\}$

$$\Rightarrow S^* = \{ j \in N \,;\, p_j \geq R^* \}$$

- Now even if we do not know R^*, it is clear that we only need to consider $S \in \{E_0, E_1, \ldots, E_N\}$

MNL with constraints

- Let $x^S \in \{0,1\}^n \equiv$ Indicator of set $S \subseteq N$
  (ie, $x^S_j \equiv \mathbb{1}\{j \in S\}$)
- We now want to solve a constrained assortment opt^n

$$\max_{x \in \{0,1\}^n} \frac{\sum_{j \in N} p_j v_j x_j}{v_0 + \sum_{j \in N} v_j x_j}$$

$$\text{s.t} \quad \sum_{j \in N} a_{ij} x_j \leq b_i \quad \forall i \in L$$

$$x_j \in \{0,1\} \quad \forall j \in N$$

- Assumption - $A = \{a_{ij}\}$ is totally unimodular, $b_i \in \mathbb{Z}$
  ($\Rightarrow$ extreme points of $\{Ax \leq b\}$ are integral)

Eg - $\sum_{j \in N} x_j \leq C$

- If $N = \underbrace{S_1 \cup S_2 \cup \ldots, S_k}_{\text{Partition}}$, $\sum_{j \in S_i} x_j \in \{b_{S_i}, \ldots, B_{S_i}\}$

- Joint pricing and assortment opt^n
  - Products $N = \{1, \ldots, n\}$, prices $P = \{p_1, p_2, \ldots, p_K\}$
  - $v_{ik} \equiv$ attractiveness of product i at price p_k
  - Idea - Create virtual products: $x_{ik} \equiv$ product i at price k
    - Constraint: at most one $x_{ik} = 1$ for every i

- How do we solve constrained MNL pricing?

$\boxed{\text{OPT1}}$: $\max \sum_{j \in N} \frac{P_j v_j x_j}{v_0 + v^T x}$

s.t. $Ax \leq b$

$(L \times N)$

$x_j \in \{0,1\}$

$\boxed{\text{OPT2}}$

$$\max \sum_{j \in N} P_j y_j$$

$$\text{s.t.} \sum_{j \in N} y_j + y_0 = 1$$

$$\sum_{j \in N} \frac{a_{ij}}{v_j} y_j \leq \frac{b_i}{v_0} y_0 \quad \forall i \in L$$

$$0 \leq \frac{y_j}{v_j} \leq \frac{y_0}{v_0} \quad \forall j \in N$$

<u>Thm</u> - The above problems have the same optimal objective. Moreover, given a solution to OPT2, we can construct a solution to OPT 1.

<u>Pf</u> - First, as in prev result, we have OPT1 is equiv to

$\boxed{\text{OPT3}}$ $\max \sum_{j \in N} (P_j - R^*) \frac{v_j}{v_0} x_j$, where $R^* \equiv$ OPT1 objective

s.t. $Ax \leq b$, $0 \leq x_j \leq 1$

This follows from MNL form + total unimodularity of A

Thus we need to show OPT3 $\equiv$ OPT2

- Let $\{y_j^*\}_{j\in N\cup\{0\}}$ be an optimal soln to OPT2
  $\{x_j^*\}_{j\in N\cup\{0\}}$ be an optimal soln to OPT3

By defn, $OPT3(x_j^*) = R^*$

- Now we show $\gamma^* \triangleq OPT2(y_j^*) = R^*$

Let $\hat{y}_j = \dfrac{v_j x_j^*}{v_0 + \sum v_i x_i^*}$, $\hat{y}_0 = 1 - \sum_{j\in N} \hat{y}_j = \dfrac{v_0}{v_0 + \sum v_i x_i^*}$

then $\{\hat{y}_i\}$ satisfies constraints of OPT2

- $\displaystyle\sum_{j\in N} \frac{a_{ij}}{v_j}\hat{y}_j = \sum_{j\in N} \frac{a_{ij}\, x_j^*}{v_0 + \sum v_i x_i^*} \le \frac{b_i}{v_0 + \sum v_i x_i^*} = \frac{b_i \hat{y}_0}{v_0} \;\forall i$

- $\dfrac{\hat{y}_j}{v_j} = \dfrac{x_j^*}{v_0 + \sum v_i x_i^*} \le \dfrac{\hat{y}_0}{v_0} \quad \forall j$

$\Rightarrow \{\hat{y}_i\}$ is feasible for OPT2

$\Rightarrow \gamma^* \ge OPT2(\{\hat{y}_i\}) = \displaystyle\sum_j P_j \frac{x_j^* v_j}{v_0 + \sum v_i x_i^*} = R^*$

- Now suppose $\gamma^* = OPT2(\{y_i^*\}) > R^*$. Note $y_0^* > 0$

Consider $\hat{x}_j = \dfrac{y_j^*/v_j}{y_0^*/v_0}$ – Check that $\hat{x}_j$ is feasible for OPT 3

Then $OPT3(\{\hat{x}_j\}) = \displaystyle\sum_{j\in N} (P_j - R^*)\frac{v_j}{v_0}\hat{x}_j = \frac{1}{y_0^*}\sum_{j\in N} P_j y_j^* - \frac{R^*}{y_0^*}(1-y_0^*)$

$> \gamma^*/y_0^* - \gamma^*(1-y_0^*)/y_0^* = \gamma^*$ $\Rightarrow$ contradiction