## Problem 1: Understanding the Network RM Value Function

An airline operates a flight network among 3 locations A, B and C, as shown in the figure below.

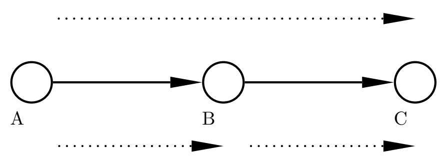


Figure 1: Two-leg airline network, with flights $A \to B$ and $B \to C$. We want to sell itineraries $(A, B)$, $(B, C)$ and also $(A, C)$; the latter requires reserving one seat on each leg.

Assume that we can have at most one itinerary request at each time period. A part of the value function for time period $t + 1$ has been computed and given in the following table.

| Remaining Capacity for B-C→ | | | | |
|---|---|---|---|---|
| Remaining Capacity for A-B ↓ | 0 | 1 | 2 | 3 |
| 0 | 0 | 600 | 900 | 1000 |
| 1 | 300 | 700 | 1000 | 1300 |
| 2 | 400 | 800 | 1100 | 1300 |
| 3 | 500 | 850 | 1100 | 1300 |

For example, if the remaining capacity on flight leg A-B is 2 and the remaining capacity on flight leg B-C is 1, then the value of the value function at time period $t + 1$ is $V_{t+1}([2, 1]) = 800$.

### Part (a)

Assume that the remaining capacity on flight leg A-B is 1 and the remaining capacity on flight leg B-C is 2 at time period $t$. Suppose we get a request for a $A - B$ flight for \$150. Is it optimal to accept a this request? Why?

**Solution:** Recall that we accept a request (for 1 seat) for any product $j$ at time period $t$ if:

1. We have enough remaining capacity on all resources which are required by product $j$, and
2. $p_j \geq V_{t+1}(\underline{\mathbf{x}}) - V_{t+1}(\underline{\mathbf{x}} - A_j)$, where $\underline{\mathbf{x}}$ is the vector of current remaining capacities (i.e., state of the system) and $A_j = \{a_{ij}\}_{\{i \in [m]\}}$ is the $j^{th}$ column of the incidence matrix $A$, which contains the resource requirement $a_{ij}$ of product $j$ for each resource $i$.

For this part, we have that $\underline{\mathbf{x}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and the arriving request is for a product $j$ with $p_j = 150$ and $A_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. From the table, we have $V_{t+1}(\underline{\mathbf{x}}) = 1000$ and $V_{t+1}(\underline{\mathbf{x}} - A_j) = V_{t+1}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) = 900$ and hence $V_{t+1}(\underline{\mathbf{x}}) - V_{t+1}(\underline{\mathbf{x}} - A_j) = 100 < 150$; thus we should accept the request.

### Part (b)

Assume that the remaining capacity on flight leg A-B is 2 and the remaining capacity on flight leg B-C is 3 at time period $t$. Is it optimal to accept a request for a $A - C$ itinerary for \$275 at time period $t$? Why?

**Solution:** For this part, we have $\underline{\mathbf{x}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and the arriving request is for a product $k$ with $p_k = 275$ and $A_j = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. From the table, we have $V_{t+1}(\underline{\mathbf{x}}) = 1300$ and $V_{t+1}(\underline{\mathbf{x}} - A_j) = V_{t+1}\left(\begin{pmatrix} 0 \\ 2 \end{pmatrix}\right) = 1000$ and hence $V_{t+1}(\underline{\mathbf{x}}) - V_{t+1}(\underline{\mathbf{x}} - A_k) = 300 > 275$; thus we should reject the request.

## Problem 2: LP-based Allocations and Bid-prices for Network RM

In this problem, we will implement the LP-based approximation methods we studied in class to derive allocation decisions for a simple network.

We want to sell tickets on two flights, Flight 1 from San Francisco (SFO) to Detroit (DIA), and Flight 2 from DIA to St. Louis (STL); a person flying from SFO to STL needs a seat on both flights. Consider selling tickets over 300 time periods (for example, think of 60 days, with 5 time periods per day), with initial capacity of 100 on Flight 1 (SFO-DIA), and 120 on Flight 2 (DIA-STL). The airlines offers a total of 6 products. In particular, for each origin-destination (OD) pair, there are two fare classes – class Y corresponding to the full-fare economy class ticket, and class M corresponding to the discount-fare economy class ticket. In each period, the airlines gets at most one request for a particular product, where product $j$ is requested with probability $\lambda_j$. The fares and per-period request probabilities for the products are given in the following table. Using

| Product | Class | Total Fare | Per-period Arrival Probability |
|---|---|---|---|
| SFO-DIA | Y | \$150 | 1/10 |
| SFO-DIA | M | \$100 | 1/5 |
| DIA-STL | Y | \$120 | 1/15 |
| DIA-STL | M | \$80 | 4/15 |
| SFO-STL | Y | \$250 | 1/10 |
| SFO-STL | M | \$170 | 2/15 |

### Part (a)

Let $D_0$ be the number of time-periods in which no request arrived, and $D_j$ be the (random) demand for product $j$. What is the distribution, and average value of $D_0$? What about for $D_1$?

**Solution:** Note that the sum over all the product types of the probability that a request at any time is of that type is $1/10 + 1/5 + 1/15 + 4/15 + 1/10 + 2/15 = 13/15$; hence, there is a $2/15$ probability that a request arriving in any period $t$ corresponds to a null request. Since there are

300 time periods, and each period is independent, we have that $D_0 \sim$BINOMIAL$(300, 2/15)$, and $\mathbb{E}[D_0] = 40$. Similarly, we have $D_1 \sim$BINOMIAL$(300, 1/10)$, $\mathbb{E}[D_1] = 30$ (and so on for the rest).

### Part (b)

Let $V_t(\underline{\mathbf{x}})$ be the value function at beginning of period $t$ with state $\underline{\mathbf{x}}$. Write down a linear program that computes the *randomized-LP upper bound* $V_{300}^{UB}(\underline{\mathbf{x}})$, as well as the *fluid upper bound* $V_{300}^{FL}([100, 120])$. How do $V_{300}(\underline{\mathbf{x}})$, $V_{300}^{UB}(\underline{\mathbf{x}})$ and $V_{300}^{FL}(\underline{\mathbf{x}})$ relate to each other?

**Solution:** Let the products be numbered as $\{1, 2, \ldots, 6\}$, from top to bottom in the table. Suppose we are given $(D_1, \ldots, D_6)$, the realizations of the total demand for each product. Let $y_j$ denote the number of requests of type $j$ that we accept. Then for any state $\underline{\mathbf{x}} = \begin{pmatrix} x_{SFO-DIA} \\ x_{DIA-STL} \end{pmatrix}$, we can write the conditional randomized-LP upper bound $V_{300}^{UB}(\underline{\mathbf{x}}|\{D_1, \ldots, D_6\})$ as:

$$\begin{array}{rrll} V_{300}^{UB}(\underline{\mathbf{x}}|\{D_1, \ldots, D_6\}) : \text{Maximize} & 150 \cdot y_1 + 100 \cdot y_2 + 120 \cdot y_3 & +80 \cdot y_4 + 250 \cdot y_5 + 170 \cdot y_6 & \\ \text{Subject to} & y_1 + y_2 + y_5 + y_6 & \leq x_{SFO-DIA} & \\ & y_3 + y_4 + y_5 + y_6 & \leq x_{DIA-STL} & \\ & y_j & \leq D_j & \forall j \in \{1, 2, \ldots, 6\} \\ & x_j & \geq 0 & \forall j \in \{1, 2, \ldots, 6\} \end{array}$$

and we define $V_{300}^{UB}(\underline{\mathbf{x}}) = \mathbb{E}[V_{300}^{UB}(\underline{\mathbf{x}}|\{D_1, \ldots, D_6\})]$.
Next, let $\mu_j = \mathbb{E}[D_j]$ for each $j \in \{1, 2, \ldots, 6\}$; note that from the table, we have $(\mu_1, \mu_2, \ldots, \mu_6) = (30, 60, 20, 80, 30, 40)$. Then the fluid upper bound is given by:

$$\begin{array}{rrll} V_{300}^{FL}(\underline{\mathbf{x}}) : \text{Maximize} & 150 \cdot y_1 + 100 \cdot y_2 + 120 \cdot y_3 & +80 \cdot y_4 + 250 \cdot y_5 + 170 \cdot y_6 & \\ \text{Subject to} & y_1 + y_2 + y_5 + y_6 & \leq x_{SFO-DIA} & \\ & y_3 + y_4 + y_5 + y_6 & \leq x_{DIA-STL} & \\ & y_j & \leq \mu_j & \forall j \in \{1, 2, \ldots, 6\} \\ & x_j & \geq 0 & \forall j \in \{1, 2, \ldots, 6\} \end{array}$$

Finally, recall from the lecture that we have:

$$V_{300}(\underline{\mathbf{x}}) \leq V_{300}^{UB}(\underline{\mathbf{x}}) \leq V_{300}^{FL}(\underline{\mathbf{x}})$$

### Part (c)

Solve the fluid upper-bound LP to compute approximate seat allocations for each class, and the fluid upper bound $V_{300}^{FL}([100, 120])$
*Hint: You should write a function to solve the LP given the RHS values of the constraints; you will need to use this in the next few parts*

**Solution:** Substituting $x_{SFO-DIA} = 100$ and $x_{DIA-STL} = 120$ in the above fluid LP, we get that the optimal seat allocations are given by $(y_1, y_2, \ldots, y_6) = (30, 40, 20, 70, 30, 0)$, and the corresponding fluid upper bound is $V_{300}^{FL}([100, 120]) = \$24,000$.

### Part (d)

Assume that the $\{D_j\}$, the *total* demand (i.e., total number of requests over time) for each product $j$ is an independent random variables with the distribution as in part $a$. Find the randomized-LP upper bound $V_{300}^{UB}([100, 120])$ via simulation in this case, and compare with the fluid bound.

### Part (e)

Write down the dual of the fluid upper-bound LP, and use it to compute bid prices for each flight.

**Solution:** Consider the fluid LP given above. Let $\beta_j$ be the dual variable corresponding to each constraint of the form $y_j \leq \mu_j$, and $z_i$ be the dual variable corresponding to each constraint of the form $\sum_j a_{ij} y_j \leq x_i$. Then the dual of the LP is given by

$$\begin{array}{lrll} \text{Minimize} & 100 \cdot z_{SFO-DIA} + & 120 \cdot z_{DIA-STL} + \sum_j \mu_j \beta_j & \\ \text{Subject to} & \beta_j + z_i & \leq p_j & \forall j \in \{1, 2, \ldots, 6\} \\ & \beta_j & \geq 0 & \forall j \in \{1, 2, \ldots, 6\} \\ & z_i & \geq 0 & \forall i \in \{SFO-DIA, DIA-STL\} \end{array}$$

Solving, we get $z_{SFO-DIA} = 100$ and $z_{DIA-STL} = 80$. These are thus the *bid-prices* for seats on each corresponding flight, i.e., we accept a SFO-DIA request only if the price is $\geq 100$, a DIA-STL request only if the price is $\geq 80$, and a SFO-STL request only if the price is $\geq 180$.

### Part (f)

Next, generate sample arrival sequences, and find the average revenue earned if we admit product requests according to $(i)$ admit all feasible incoming request, $(ii)$ booking limits for each product class as computed via the fluid LP in part (d), and $(iii)$ the bid-price policy (i.e., where we admit products whose price is greater than the sum of bid-prices of its required resources) from part (e).

### Part (g)

Next, suppose the system becomes twice as busy; we now have 10 periods per day, with at most one arrival every period (and thus we have 600 time periods in 60 days) with the same arrival probabilities as before. In response, the airline announces two flights from SFO to DIA (total capacity 200), and two flights from DIA to STL (total capacity 240). Argue that the bid-prices you compute in part (e) *remain the same* in this setting. What is the new fluid upper-bound?

**Solution:** We now want to find $V_{600}^{FL}(2\underline{\mathbf{x}})$; as before, the dual of the fluid LP is given by

$$\begin{array}{lrll} \text{Minimize} & 2 \times 100 \cdot z_{SFO-DIA} + & 2 \times 120 \cdot z_{DIA-STL} + \sum_j 2 \times \mu_j \beta_j & \\ \text{Subject to} & \beta_j + z_i & \leq p_j & \forall j \in \{1, 2, \ldots, 6\} \\ & \beta_j & \geq 0 & \forall j \in \{1, 2, \ldots, 6\} \\ & z_i & \geq 0 & \forall i \in \{SFO-DIA, DIA-STL\} \end{array}$$

Note that this is identical to part ($e$), except that all the coefficients in the objective are multiplied by 2; consequently, the bid prices remain the same, and moreover, $V_{600}^{FL}(2\underline{\mathbf{x}}) = 2 \times V_{300}^{FL}(\underline{\mathbf{x}})$.

### Part (h)

We now want to see how our bid-price policy does when the system becomes bigger. Let $R^{(k)}$ denote the average revenue, and $V^{FL,(k)}$ denote the fluid upper bound, in the system where all capacities and all demands are scaled by $k$ (for example, in part g, we had $k = 2$). Repeat the simulation from part (f) with capacities and demands increased by a factor of $k = 2, 4, 6, 8, 10$, and in each case, plot $\frac{R^{(k)} - V^{FL,(k)}}{V^{FL,(k)}}$.

### Part (i)

(Optional) Repeat part (f), but now recompute the bid-prices at the start of each day.