Concavity and random LPs

- In the process of bounding the revenue in the single-resource allocation problem, we obtained the following LP

$$V_n^{UB}(c \mid \{D_1, D_2, \ldots, D_n\}) \equiv \max \sum_{i=1}^{n} p_i x_i$$
$$\text{s.t.} \sum_{i=1}^{n} x_i \le c$$
$$x_i \le D_i \ \forall i$$
$$x_i \ge 0 \ \forall i$$

upper bound given random demands $D_1, D_2, \ldots, D_n$

This is sometimes called the randomized-LP bound

- We also defined the fluid-LP bound

$$V_n^{FL}(c) \equiv \max \sum_{i=1}^{n} p_i x_i$$
$$\text{s.t. } x_i \le \mu_i \quad (\mu_i = \mathbb{E}[D_i])$$
$$\sum_{i=1}^{n} x_i \le c$$
$$x_i \ge 0 \ \forall i$$

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Finally we showed $\boxed{V_n^{Fl}(c) \geq \mathbb{E}\left[V_n^{UB}(c \mid \{D_i\}_i)\right]}$

We will now see how to derive such results directly using convexity and Jensen's Inequality.

Main tools from convexity

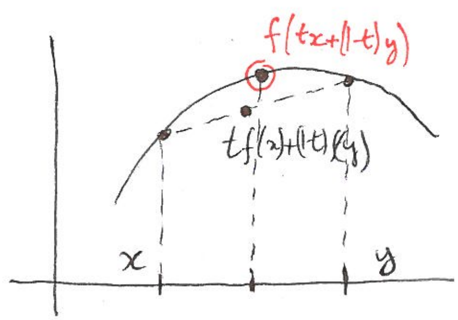
- **Defn** - A function $f(x)$ is convex if $\forall\, x, y$ and $\forall\, t \in [0,1]$, the function obeys

$$(*) \quad \boxed{f\left(tx + (1-t)y\right) \leq t f(x) + (1-t) f(y)}$$

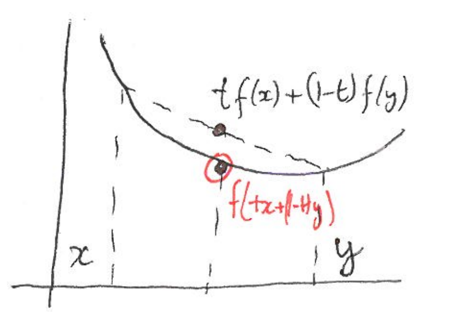
- **Notes** -
  i) Function $f(x)$ is concave if $-f(x)$ is convex (alternately, replace $\leq$ with $\geq$ in $(*)$ to get concavity)

  ii) The definition works even if x is a vector

  iii) Graphical way to remember.



- suppose $t = 1/2$
- $tx + (1-t)y$ is the pt midway bet^n x, y



- Operations which preserve convexity (See HW2)

  i) Scaling - $f(x)$ convex, $a \geq 0$, then
  $$f(ax+b) \text{ is convex}$$

  ii) Linear Combination - $f_1(x)$, $f_2(x)$ convex, $a_1, a_2 \geq 0$
  $$\Rightarrow \quad a_1 f_1(x) + a_2 f_2(x) \text{ is convex}$$

  iii) Maximization - $f_1(x), f_2(x)$ convex, then
  $$\max\big(f_1(x), f_2(x)\big) \text{ is convex}$$

(there are many others, but these suffice for us)

- For concavity, it is preserved under scaling, linear combination and minimization.

- A linear function $f(x) = ax+b$ is both convex and concave.

- Jensen's Inequality - If $f(x)$ is convex, then
$$E[f(x)] \underset{\left(\underset{\text{concave}}{\leq}\right)}{\overset{\text{convex}}{\geq}} f(E[x])$$

Now we show how to use these for LPs

- Primal Argument

  - We want to show that the expected value of the randomized LP is bounded by the fluid LP.

$$\mathbb{E}\left[\begin{array}{ll} \max & \sum_{i=1}^{n} p_i x_i \\ \text{s.t} & \sum_{i=1}^{n} x_i \le c \\ & x_i \le D_i \quad \forall i \\ & x_i \geq 0 \quad \forall i \end{array}\right] \le \left[\begin{array}{ll} \max & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} x_i \le c. \\ & x_i \le \mathbb{E}[D_i] \quad \forall i \\ & x_i \geq 0 \end{array}\right] \qquad (**)$$

  - To see that this can be shown via Jensen's, we need to first understand that the optimization problem above is a function ~~f~~ g which maps 'capacities' $(D_1, D_2, \ldots, D_n)$ to an output. Visualize this as follows

$$(d_1, d_2, \ldots, d_n) \longrightarrow \boxed{\begin{array}{ll} y = \max & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} & \sum_{i=1}^{n} x_i \le c \\ & x_i \geq 0 \ \forall i \\ & x_i \le d_i \ \forall i \end{array}} \longrightarrow y = \cancel{f}\, g(d_1, d_2, \ldots, d_n)$$

$$\cancel{f} \quad g$$

Thus, if we show $g(d_1, d_2, \ldots, d_n)$ is concave in $(d_1, d_2, \ldots, d_n)$, we can use Jensen's Inequality to get (**)

- Suppose we have 2 inputs

$$\underline{d} = (d_1, d_2, \ldots, d_n)$$
$$\underline{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$$

Let

$$\underline{x}_d^* \equiv \arg\max \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i \leq c, \quad 0 \leq x_i \leq d_i \ \forall i$$

$$\underline{x}_\delta^* \equiv \arg\max \sum_{i=1}^{n} p_i x_i \quad \text{s.t.} \quad \sum_{i=1}^{n} x_i \leq c, \quad 0 \leq x_i \leq \delta_i$$

(and $g(\underline{d})$, $g(\underline{\delta})$ the corresponding objective values)

- For some $t \in [0,1]$, consider $\underline{\Delta} = t\underline{d} + (1-t)\underline{\delta}$ (i.e., for each i, $\Delta_i = t d_i + (1-t)\delta_i$)

To show concavity, we need to show

$$g(\underline{\Delta}) \geq t\, g(\underline{d}) + (1-t)\, g(\underline{\delta})$$

- To see this, we first observe that the point $x_\Delta = t\,x_d^* + (1-t)\,x_\delta^*$ is feasible for demand vector Δ, i.e

i) $$\sum_{i=1}^{n} x_{\Delta,i} = t\sum_{i=1}^{n} x_{d,i}^* + (1-t)\sum_{i=1}^{n} x_{\delta,i}^* \leq tc + (1-t)c = c$$

ii) $$x_{\Delta,i} = t\,x_{d,i}^* + (1-t)\,x_{\delta,i}^* \leq t\,d_i + (1-t)\,\delta_i = \Delta_i$$

$$\Rightarrow \sum_{i=1}^{n} p_i\, x_{\Delta,i} \leq \left[\begin{array}{l} \max\limits_{\underline{x}} \sum p_i x_i \\ \text{s.t} \quad \sum x_i \leq c \\ \qquad 0 \leq x_i \leq \Delta_i \ \forall i \end{array}\right] = g(\underline{\Delta})$$

However $$\sum_{i=1}^{n} p_i\, x_{\Delta,i} = t\sum_{i=1}^{n} p_i\, x_{d,i}^* + (1-t)\sum_{i=1}^{n} p_i\, x_{\delta,i} = t\,g(\underline{d}) + (1-t)\,g(\underline{\delta})$$

Thus $g(\underline{\Delta}) \geq t\,g(\underline{d}) + (1-t)\,g(\underline{\delta}) \Rightarrow g$ is concave.

Dual argument

We will now see an alternate (easier?) way to see the same result via duality

- We need one fact for this: Earlier, we said that if f_1, f_2 are convex, then $\max(f_1(x), f_2(x))$ is convex. Similarly, if g_1, g_2 are concave fns, then $\min(g_1(x), g_2(x))$ is concave. (try to check this!)
- Now, by LP duality, we have

$$g(d_1, \dots, d_n) = \left[\begin{array}{ll} \max \sum_{i=1}^{n} p_i x_i & \\ \text{s.t } \sum_{i=1}^{n} x_i \leq c & : z \\ 0 \leq x_i \leq d_i \ \forall i & : \beta_i \end{array}\right] \begin{pmatrix}\text{dual}\\ \text{vars}\end{pmatrix}$$

PRIMAL

$$= \left[\begin{array}{l} \min \ cz + \sum_{i=1}^{n} d_i \beta_i \\ \text{s.t } \beta_i + z \geq p_i \ \forall i \\ \beta_i \geq 0, \ z \geq 0 \end{array}\right]$$

- Now let $\mathcal{X}$ be the set of extreme points of the dual constraint set, i.e.

$$\mathcal{X} = \left\{ (\underline{\beta}, z) \;\middle|\; (\underline{\beta}, z) \text{ is an extreme pt of } \begin{array}{l} \beta_i + z \geq p_i \ \forall i \\ \beta_i \geq 0,\ z \geq 0 \end{array} \right\}$$

Then
$$g(d_1, \dots, d_n) = \min_{(\underline{\beta}, z) \in \mathcal{X}} \Big\{ \underbrace{cz + \sum_{i=1}^{n} d_i \beta_i}_{\text{linear (hence concave) in } (d_1, d_2, \dots, d_n)} \Big\}$$

$\Rightarrow$ $\underline{g(d_1, \dots, d_n) \text{ is concave in } d_1, \dots, d_n}$

Thus - $$\underbrace{\mathbb{E}\left[g(D_1, D_2, \dots, D_n) \right]}_{\mathbb{E}\left[\begin{array}{ll} \max & \sum_{i=1}^{n} x_i p_i \\ \text{s.t} & \sum x_i \leq c \\ & 0 \leq x_i \leq D_i \end{array}\right]} \leq \underbrace{g(\mathbb{E}[D_1], \dots, \mathbb{E}[D_n])}_{\left[\begin{array}{ll} \max & \sum_{i=1}^{n} x_i p_i \\ \text{s.t} & \sum x_i \leq c \\ & 0 \leq x_i \leq \mathbb{E}[D_i] \end{array}\right]}$$