Example of Protection Level Computation

Consider a single-resource capacity allocation problem with:

- 3 demand classes and $C = 4$ units of capacity.
- The fare for each class is $p_1 = 10$, $p_2 = 5$, and $p_3 = 4$.
- The demand distribution for each class is given in the following table.

d	$\mathbb{P}\{D_1 = d\}$	$\mathbb{P}\{D_2 = d\}$	$\mathbb{P}\{D_3 = d\}$
0	1/9	1/5	2/7
1	2/9	1/5	1/7
2	3/9	1/5	2/7
3	2/9	1/5	1/7
4	1/9	1/5	1/7

For $k \in \{3, 2, 1\}$, let $V_j(s)$ denote the value function in stage k, that is, the maximum expected revenue that can be obtained from class $k, k-1, \ldots, 1$, given that we have s units of capacity remaining at the beginning of stage k. We now compute the value functions and protection levels.

- First we compute $V_1(s) = 10.\mathbb{E}[\min\{s, D_1\}]$. We have:

$$V_1(0) = 0$$

$$V_1(1) = 10 \cdot \mathbb{P}[D_1 \geq 1] = \frac{80}{9}$$

$$V_1(2) = 10 \cdot (1 \cdot \mathbb{P}[D_1 = 1] + 2 \cdot \mathbb{P}[D_1 \geq 2]) = 10 \cdot \left(1 \cdot \frac{2}{9} + 2 \cdot \frac{6}{9}\right) = \frac{140}{9}$$

  Similarly we can compute the rest to get the following table: Observe that this is

x	0	1	2	3	4
$V_1(x)$	0	80/9	140/9	170/9	180/9

  increasing and concave (i.e., $\Delta V_1(s) = V_1(s+1) - V_1(s)$ is decreasing).

- Next we compute $h_2(y) = -p_2 \cdot y + V_1(y)$. This gives Note that this also is concave.

x	0	1	2	3	4
$h_2(x)$	0	35/9	50/9	35/9	0/9

  Moreover, the protection level is given by $x_1^* = \max_y\{h_1(y)\} = 2$

- You can also find this using Littlewood's rule:

$$x_1^* = \max_{y \in \mathbb{N}} \left[\mathbb{P}[D_1 \geq y] > p_2/p_1 = 1/2\right]$$

  Plot $\mathbb{P}[D_1 \geq y]$ and check that the largest value where this is greater than 1/2 is 2.

- For selling tickets of fare class 2, note that we want to protect 2 seats for fare-class 1. Thus

$$V_2(0) = V_1(0), V_2(1) = V_1(1), V_2(2) = V_1(2)$$

  . For the remaining, we have

$$V_2(3) = \mathbb{P}[D_2 = 0] \cdot (V_1(3)) + \mathbb{P}[D_2 \geq 1] \cdot (5 + V_1(2)) = \frac{182}{9}$$
$$V_2(4) = \mathbb{P}[D_2 = 0] \cdot (V_1(4)) + \mathbb{P}[D_2 = 1] \cdot (5 + V_1(3)) + \mathbb{P}[D_2 = 1] \cdot (5 \cdot 2 + V_1(2)) = \frac{217}{9}$$

  Thus we get the following table: Observe that this is increasing and concave, and also

x	0	1	2	3	4
$V_1(x)$	0	80/9	140/9	182/9	217/9

  $V_2(s) \geq V_1(s)$ for all s (why is this intuitive?).

  A more subtle thing to check is that $\Delta V_2(s) \geq \Delta V_1(s)$ – in words, adding an extra seat at an earlier stage can bring a higher value than adding it at a later stage. This should seem plausible via a *simulation argument*: we could always hold that seat back for the later stage, instead of using it optimally at the current stage.

- As before, we compute $h_3(y) = -p_3 \cdot y + V_2(y)$. This gives Again note that this also is

x	0	1	2	3	4
$h_2(x)$	0	44/9	68/9	74/9	73/9

  concave. Moreover, the next protection level is given by $x_2^* = \max_y \{h_2(y)\} = 3$

- Note that the protection levels are non-decreasing, i.e., $x_1^* \leq x_2^* \leq \ldots$.

- In terms of finding the optimal policy, we are done! However, we can find $V_3(\cdot)$ to find the expected revenue $V_3(4)$. As before, $V_3(s) = V_2(s)$ whenever $s \leq x_2^* = 3$. Finally we have

$$V_3(4) = \mathbb{P}[D_3 = 0]V_2(4) + \mathbb{P}[D_3 \geq 1] \left(V_2(3) + 4\right) = \frac{508}{21}$$