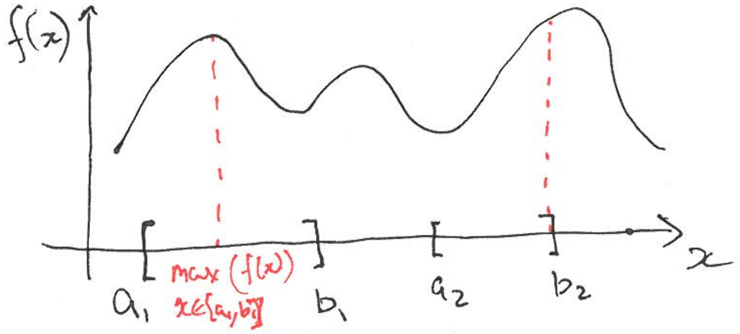
①

$$\hat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+, \ldots, s_k\}} \{ -P_k y_k + \hat{V}_{k-1}(y_k) \right]$$

$$\text{where} \quad \hat{V}_k(s_k) = \mathbb{E}\left[\hat{V}_k(s_k \mid D_k)\right]$$

- Consider $Y = \max_{x \in [A,B]} \{ f(x) \}$

random variable ↗ (Y) ← fixed function (f) ← random variables, $A \leq B$

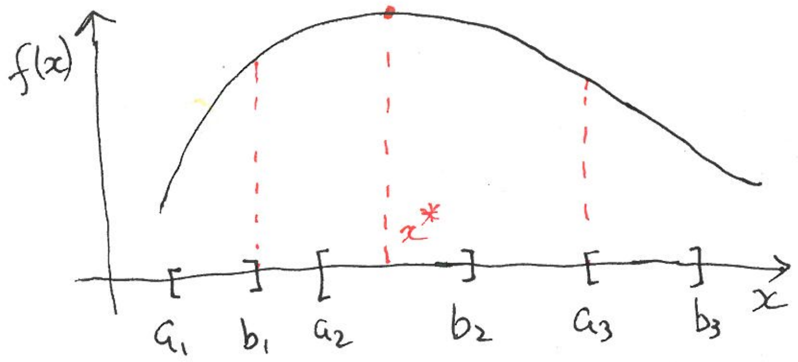


- For general f, this is some complicated random variable

- Suppose f is concave. Let $x^* = \arg\max f(x)$

Now Y is easy to define

$$Y(A,B) = \begin{cases} B & ; B < x^* \\ x^* & ; A \leq x^* \leq B \\ A & ; A > x^* \end{cases}$$



- Now consider $h_k(y) = -p_k\, y + \hat{V}_{k+1}(y)$

Then $\hat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y \in \{(s_k - D_k)^+, \dots, s_k\}} h_k(y) \right]$

- Suppose $h_k(\cdot)$ is concave, $x^*_{k-1} = \arg\max_{y \in [0,c]} h_k(y)$

Then
$$y^*_k = \begin{cases} s_k & ; \ s_k < x^*_{k-1} \\ x^*_{k-1} & ; \ (s_k - D_k)^+ \leq x^*_{k-1} \leq s_k \\ (s_k - D_k)^+ & ; \ (s_k - D_k)^+ > x^*_{k-1} \end{cases}$$

- This is a protection level policy (with protection level x^*_{k-1})

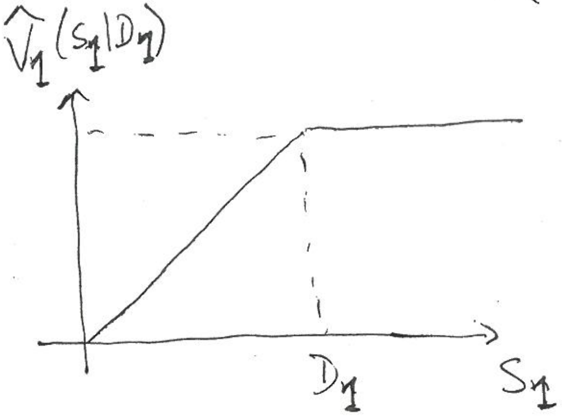
i.e, at each stage k, for all states s_k and demand D_k; we do the following

1) If $s_k < x^*_{k-1}$ (i.e, already below protection level), then do not admit anyone
2) If $(s_k - D_k)^+ \leq x^*_{k-1} \leq s_k$ (i.e., excess demand), then admit up to protection level x^*_{k-1}
3) If $(s_k - D_k)^+ > x^*_{k-1}$ (ie., excess supply), then admit all customers D_k up to capacity limits

Is $h_k(x)$ concave?

- $$\hat{V}_1(s_1|D_1) = \max_{(s_1-D_1)^+ \le y \le s_1} \left[P_1 s_1 - \underbrace{P_1 (s_1 - D_1)^+}_{} \right]$$

i.e, $P_1 \cdot \min\{s_1, D_1\}$



$*$ $\hat{V}_1(s_1|D_1)$ concave in s_n

$*$ $\hat{V}_1(s_1) = \mathbb{E}[\hat{V}_1(s_1|D_1)]$ is concave in s_1 (linear combination of concave fns)

Moreover $$h_2(y) = \underbrace{-P_2 y}_{\text{linear}} + \underbrace{\hat{V}_1(y)}_{\text{concave}} \Rightarrow \text{concave!}$$

- Now we prove $h_k(y)$ (and $\hat{V}_k(y)$) are concave by induction

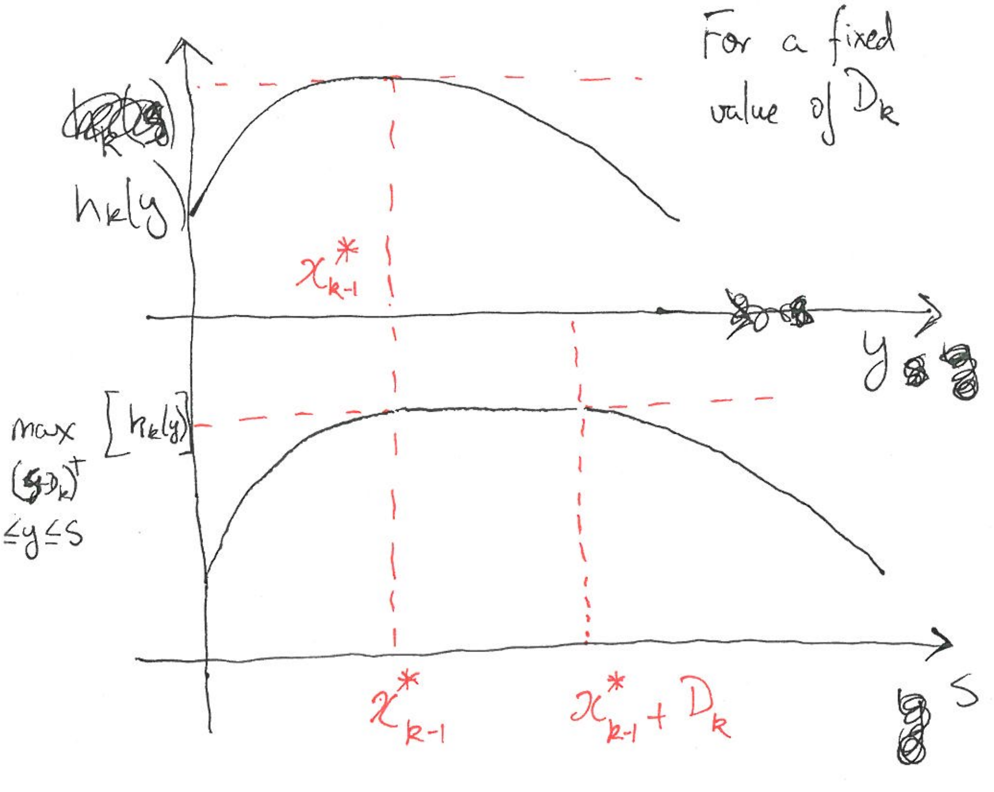
  - Assume $\hat{V}_{k-1}(y)$ is concave in y
  - $h_k(y) = -P_k y + V_{k-1}(y) \Rightarrow$ concave
  - $$\hat{V}_k(s|D_k) = \underbrace{\max_{(s-D_k)^+ \le y \le s} [h_k(y)]}_{\text{is this convex?}} + \underbrace{P_k s_k}_{\text{linear}}$$

- Let $\boxed{x^*_{k-1} = \underset{y \in [0,\infty)}{\arg\max} \left[h_k(y) \right]}$

$$\left(\text{Note : } x_1^* = F_1^{-1}\left(1 - \frac{P_2}{P_1}\right) \text{ - Littlewood's Rule!} \right)$$

(Check this by writing out $h_2(y)$, and comparing to first class)

- We know $h_k(y)$ is convex, so we can maximize it over $(s-D_k)^+ \le y \le s$



- $$y_k^* = \underset{y \in [(s-D_k)^+, s]}{\arg\max} \left[h_k(y) \right]$$

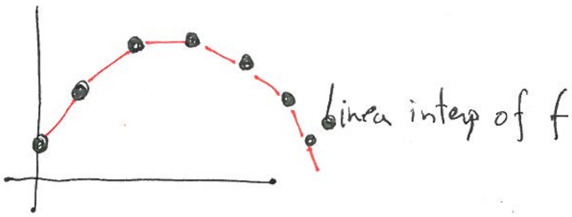
$$= \begin{cases} s & ; s \le x^*_{k-1} \\ x^*_{k-1} & ; s-D_k < x^*_{k-1} \le s \\ s - D_k & ; s - D_k \ge x^*_{k-1} \end{cases}$$

- Thus, $\max\limits_{(s-D_k)^+ \le y \le s} \left[h_k(y) \right]$ is concave in s

$$\Rightarrow \hat{V}_k(s \mid D_k) \text{ is concave} \Rightarrow \hat{V}_k(s) \text{ is concave}$$

- The above argument also works when D_j are discrete

Main Idea - Consider the linear interpolation of the discrete fns $V_k(s)$



- this is concave for $\hat{V}_1(s_1|D_1)$
- the rest of the argument is identical

- In particular, the argument gives

$$\forall s \quad \boxed{\Delta \hat{V}_k(s) = \hat{V}_k(s+1) - \hat{V}_k(s) \leqslant \Delta \hat{V}_k(s-1)}$$

  - Diminishing returns to revenue from increasing capacity

- Finally, observe that the optimal policy given D_k is to accept as many customers as we can till # of remaining seats $= x^*_{k-1}$ (protection level for classes $k-1, k-2, \ldots, 1$)
  - This can be implemented without knowing D_k !!