ORIE 4154/5154: Pricing and Market Design **Fall 2026**

Lecture 20 — Myerson's Lemma and the Payment Identity

Lecturer: Sid Banerjee *Date: November 3, 2026 (draft)*

20.1 What if service is gradual or randomized?

VCG in Lecture 18 made welfare maximization truthful, even for general valuation functions. Lecture 19 then characterized deterministic single-parameter win/lose allocations, including inefficient ones. Holding other reports fixed, a monotone allocation changes from 0 to 1 at a threshold, and the winner pays that threshold. Now suppose the probability of receiving service rises gradually with the report. What replaces the critical price?

The answer is an area formula. A truthful agent's utility is the area under its allocation curve; payment is what remains of its value times allocation. We will derive this formula directly from incentive constraints, without assuming that the allocation curve is differentiable.

20.2 Model and the role of expectations

Each agent has one private nonnegative value for service. A randomized mechanism chooses a random feasible service vector $X(b)$ and random payments $P_i(b)$ at fixed reports b. Define

$$x_i(b) = \mathbb{E}[X_i(b)], \qquad p_i(b) = \mathbb{E}[P_i(b)],$$

where these expectations are only over the mechanism's randomization. Risk-neutral quasi-linear agents evaluate the expected utility $v_i x_i(b) - p_i(b)$. A deterministic mechanism is a special case.

We continue to require feasibility: the random service vector must be supported on the public feasible set $\mathcal{X} \subseteq \{0,1\}^n$. Its marginal allocation probabilities are therefore in the convex hull of $\mathcal{X}$. The payment theorem takes a feasible allocation rule as input; it does not allow arbitrary independent service lotteries that might exceed capacity.

For a randomized mechanism, DSIC here means *truthfulness in expectation over its randomization*, for every fixed vector of other reports. It is stronger to require truthfulness for every realization of the random choices, sometimes called universal truthfulness.

Fix an agent i and b_{-i} and abbreviate

$$x(v) = x_i(v, b_{-i}) \in [0,1], \quad p(v) = p_i(v, b_{-i}), \quad u(v) = vx(v) - p(v).$$

The last expression is its expected utility when it reports its true value v. DSIC is the family of inequalities

$$u(v) \geq vx(w) - p(w) = u(w) + (v - w)x(w) \quad \text{for all } v, w \geq 0. \tag{20.1}$$

20.3 Myerson's lemma

Theorem 20.1 (Single-parameter characterization). *A feasible allocation rule is implementable by DSIC payments if and only if each $x_i(v, b_{-i})$ is nondecreasing in its own report v. All implementing expected payments have the form*

$$\boxed{p_i(v, b_{-i}) = v x_i(v, b_{-i}) - \int_0^v x_i(z, b_{-i})\, dz + \kappa_i(b_{-i}),} \tag{20.2}$$

where $\kappa_i(b_{-i}) = p_i(0, b_{-i})$ does not depend on v. In particular, the normalization $p_i(0, b_{-i}) = 0$ determines the expected payment rule uniquely.

We prove necessity and sufficiency separately. The theorem concerns expected payments conditional on reports, so uniqueness does not specify every possible realized payment in a randomized implementation.

20.3.1 Necessity: monotonicity and the envelope identity

Adding (20.1) to the same inequality with v, w interchanged gives

$$(v - w)(x(v) - x(w)) \geq 0.$$

Thus x must be nondecreasing, exactly as in Lecture 19.

For $a < b$, the two incentive constraints also give the sharper bounds

$$(b - a)x(a) \leq u(b) - u(a) \leq (b - a)x(b). \tag{20.3}$$

Partition $[0, v]$ into $0 = z_0 < z_1 < \cdots < z_m = v$, and apply (20.3) on every small interval. Summing telescopes the utility differences:

$$\sum_{j=1}^{m} (z_j - z_{j-1}) x(z_{j-1}) \leq u(v) - u(0) \leq \sum_{j=1}^{m} (z_j - z_{j-1}) x(z_j). \tag{20.4}$$

These are lower and upper Riemann sums of a bounded monotone function. On an equally spaced partition their difference is

$$\frac{v}{m}\big(x(v) - x(0)\big) \leq \frac{v}{m},$$

which tends to zero. Hence

$$\boxed{u(v) = u(0) + \int_0^v x(z)\, dz.} \tag{20.5}$$

Since $u(0) = -p(0) = -\kappa$, rearranging $u(v) = vx(v) - p(v)$ gives (20.2). This proves necessity, including the integration constant. Jumps in x cause no problem for the argument.

20.3.2 Sufficiency: no report gives higher utility

Suppose x is nondecreasing and define payments by (20.2). The truthful payoff is $\int_0^v x(z)\,dz - \kappa$. Reporting w instead gives

$$vx(w) - p(w) = (v-w)x(w) + \int_0^w x(z)\,dz - \kappa.$$

If $v > w$, the difference between truthful utility and this payoff is

$$\int_w^v \big(x(z) - x(w)\big)\,dz \geq 0.$$

If $v < w$, it is

$$\int_v^w \big(x(w) - x(z)\big)\,dz \geq 0.$$

At $v = w$ it is zero. Monotonicity proves the inequalities in both cases, and κ cancels. This establishes DSIC and completes the characterization.

20.4 What the normalization buys us

Under $\kappa = 0$, truthful utility and payment are

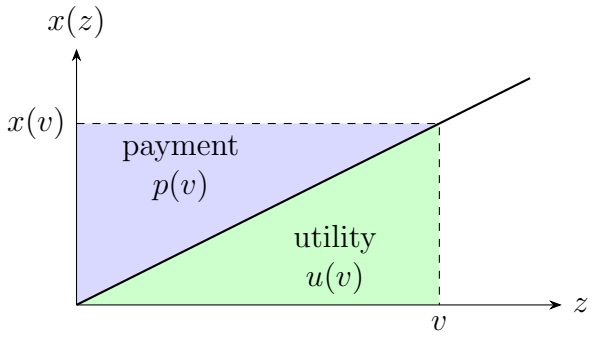
$$u(v) = \int_0^v x(z)\,dz \geq 0, \qquad p(v) = vx(v) - \int_0^v x(z)\,dz \geq 0.$$

The second inequality follows from $x(z) \leq x(v)$ on $[0, v]$. Thus normalized payments give IR conditional on every profile of other reports and no transfers from seller to agents, in expected-payment terms. Other constants preserve incentives, but a positive κ charges a zero-valued agent and violates IR at that type; a negative κ gives every type an additional subsidy.

With deterministic binary allocations, expected and realized utilities coincide. With random allocations, these guarantees refer to expectation over the mechanism's coins unless an implementation separately guarantees them in every realized outcome.

20.5 The payment formula as an area

At report v, the rectangle $vx(v)$ is the agent's expected gross value. Its truthful utility is the area below $x(z)$ between 0 and v. Payment is the area in the rectangle above that curve.



The picture uses $x(z) = z$ on $[0, 1]$, for which the two areas happen to be equal. The rectangle-minus-integral identity holds for every monotone allocation curve.

20.6 Examples

20.6.1 Thresholds, second prices, and reserves

If $x(v)$ jumps from 0 to 1 at θ, then for any winning v,

$$p(v) = v - \int_0^v x(z)\,dz = v - (v - \theta) = \theta.$$

For losing reports, payment is zero. The value of x at the single boundary point does not affect the integral, and a winner exactly at θ pays θ. Taking $\theta = \max_{j \neq i} b_j$ reproduces second-price payments; taking $\theta = \max\{r, \max_{j \neq i} b_j\}$ gives a second-price auction with reserve r.

20.6.2 A gradual allocation with an explicit lottery

Consider one agent with report domain $[0, 1]$ and allocation probability $x(v) = v$. Myerson's formula gives

$$p(v) = v^2 - \int_0^v z\,dz = \frac{v^2}{2}, \qquad u(v) = \frac{v^2}{2}.$$

An implementation draws a random threshold $Z \sim \mathrm{Unif}[0, 1]$, independent of the report, then serves the agent and charges Z exactly when its report is at least Z. At report v,

$$\mathbb{P}(Z \leq v) = v, \qquad \mathbb{E}[Z\mathbf{1}\{Z \leq v\}] = \int_0^v z\,dz = \frac{v^2}{2}.$$

For each realized threshold the procedure is a truthful posted price, so this implementation is even universally truthful and individually rational in each outcome. A realized winner pays the drawn threshold, rather than necessarily paying the unconditional mean $v^2/2$.

20.7 Ex post conditioning versus interim incentives

So far, b_{-i} has been fixed. With independent values and truthful other agents, define the *interim* allocation and payment:

$$q_i(v) = \mathbb{E}_{V_{-i}}[x_i(v, V_{-i})], \qquad t_i(v) = \mathbb{E}_{V_{-i}}[p_i(v, V_{-i})].$$

These average over other values and the mechanism's randomization. *Bayesian incentive compatibility* (BIC) requires $vq_i(v) - t_i(v) \geq vq_i(w) - t_i(w)$ for all v, w. Independence makes the distribution of V_{-i} the same for every type, so the same proof as for (20.1) gives

$$t_i(v) = vq_i(v) - \int_0^v q_i(z)\,dz + t_i(0), \tag{20.6}$$

with q_i nondecreasing. DSIC implies BIC; BIC alone need not make truthfulness optimal against every fixed profile of other reports. This distinction determines the appropriate expected-payment version of revenue equivalence in Lecture 22.

20.8 Takeaways and next question

1. VCG implements efficient allocations; monotonicity characterizes the larger family of implementable single-parameter allocations, including inefficient ones.
2. The payment identity follows from incentive inequalities and Riemann sums, including at jumps. Normalization at zero fixes the integration constant.
3. The identity can be used conditional on other reports for DSIC, or with interim functions for BIC under independent values.

Payments are now determined by allocations. Among truthful allocation rules, which one maximizes expected seller revenue? Lecture 21 will answer using the virtual values introduced in Lecture 2.

Recommended reading

- **Core:** Roughgarden, *Twenty Lectures on Algorithmic Game Theory*, Sections 3.4–3.5. We use two-report constraints, with Riemann sums covering jumps directly.
- **Course source:** Banerjee's Spring 2017 *Single-Parameter Settings and Myerson's Lemma*, especially the swapping argument and rectangle-minus-area interpretation.
- **Optional:** Karlin and Peres, *Game Theory, Alive*, Sections 14.4 and 14.6, for interim payment identities.