ORIE 4742 - Info Theory and Bayesian ML

Lecture 3: Measuring Information

February 7, 2021

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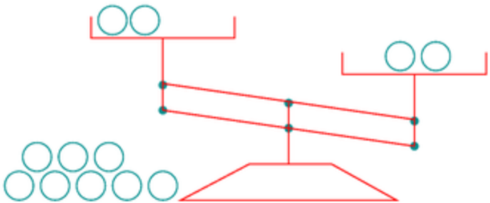
Mackay's weighing puzzle

The weighing problem



You are given 12 balls, all equal in weight except for one that is either heavier or lighter.

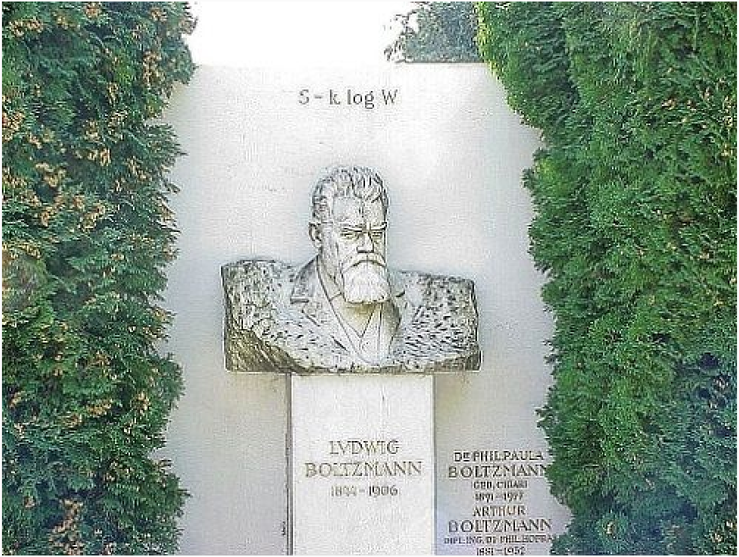
Design a strategy to determine

which is the odd ball

and whether it is heavier or lighter,

in as few uses of the balance as possible.

how much ‘information’ does a random variable have?



reading assignment: chapter 4 of Mackay

quantifying information content

measuring information

consider (discrete) rv X taking values $\mathcal{X} = \{a_1, a_2, \ldots, a_k\}$, with probability mass function $\mathbb{P}[X = a_i] = p_i \,\forall\, i, \sum_{i=1}^{k} p_i = 1$

Shannon's entropy function

- outcome $X = a_i$ has *information content*
$$h(a_i) = \log_2\left(\frac{1}{p_i}\right)$$
- random variable X has *entropy*
$$H(X) = \mathbb{E}[h(X)] = \sum_{i=1}^{k} p_i \log_2\left(\frac{1}{p_i}\right)$$

entropy: basic properties

Shannon's entropy function

- outcome $X = a_i$ has *information content*: $h(a_i) = \log_2\left(\frac{1}{p_i}\right)$
- random variable X has *entropy*: $H(X) = \mathbb{E}[h(X)] = \sum_{i=1}^{k} p_i \log_2\left(\frac{1}{p_i}\right)$

- only depends on distribution of X (i.e., $H(X) = H(p_1, p_2, \ldots, p_k)$)
- $H(X) \geq 0$ for all X
- if $X \perp\!\!\!\perp Y$, then $H(X,Y) = H(X) + H(Y)$
  where joint entropy $H(X,Y) \triangleq \sum_{(x,y)} p(x,y) \log_2 1/p(x,y)$

entropy: basic properties

Shannon's entropy function

- outcome $X = a_i$ has *information content*: $h(a_i) = \log_2\left(\frac{1}{p_i}\right)$
- random variable X has *entropy*: $H(X) = \mathbb{E}[h(X)] = \sum_{i=1}^{k} p_i \log_2\left(\frac{1}{p_i}\right)$

- if $X \sim$ uniform on $\mathcal{X}$, then $H(X) = \log_2 |\mathcal{X}|$; else, $H(X) \leq \log_2 |\mathcal{X}|$

designing questions to maximize information gain

the game of 'sixty three'

guess number $x \in \{0, 1, 2, \ldots, 62, 63\}$

designing questions to maximize information gain

the game of ’submarine’

player 1 hides a submarine in one square of an 8×8 grid

player 2 shoots at one square per round

designing questions to maximize information gain

the game of 'submarine'

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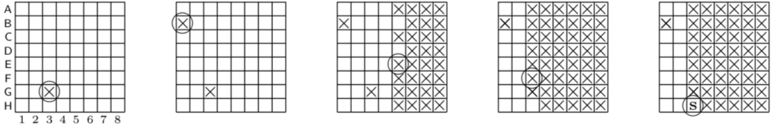
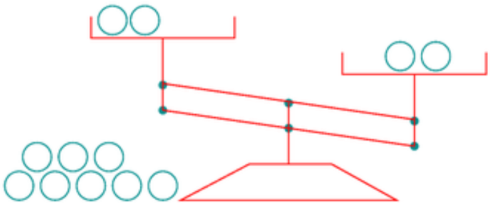


move #	1	2	32	48	49
question	G3	B1	E5	F3	H3
outcome	$x = \mathtt{n}$	$x = \mathtt{n}$	$x = \mathtt{n}$	$x = \mathtt{n}$	$x = \mathtt{y}$
$P(x)$	$\frac{63}{64}$	$\frac{62}{63}$	$\frac{32}{33}$	$\frac{16}{17}$	$\frac{1}{16}$
$h(x)$	0.0227	0.0230	0.0443	0.0874	4.0
Total info.	0.0227	0.0458	1.0	2.0	6.0

Mackay’s weighing puzzle

The weighing problem



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information acquisition in the weighing puzzle

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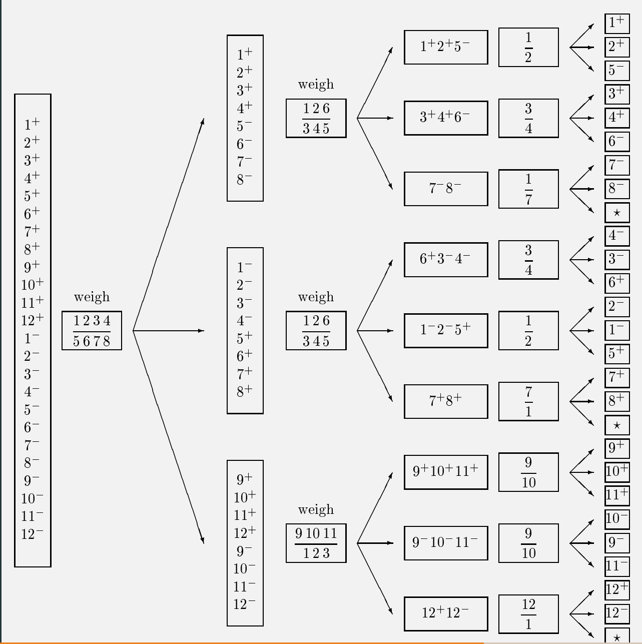
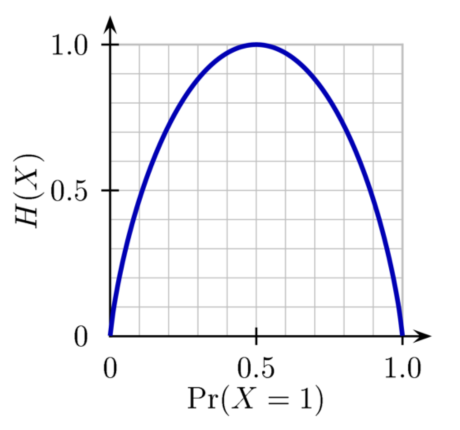
weighing game: an optimal solution



binary entropy function

if X Bernoulli(p), then $H(X) \triangleq H_2(p) = -p\log_2(p) - (1-p)\log_2(1-p)$



- (useful formula) for any $k, N \in \mathbb{N}$, $k \leq N$: $\binom{N}{k} \approx 2^{NH_2(k/N)}$

conditional entropy

suppose $X \sim \{p_1, p_2, p_3, p_4\}$, and let $Y = \mathbb{1}_{[X \in \{a_1, a_2\}]}$; then we have

$$H(X) = H(Y) + (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}\right) + (p_3 + p_4) H_2\left(\frac{p_3}{p_3 + p_4}\right)$$

conditional entropy

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conditional entropy

for any rvs X, Y: $H(X|Y) = \sum_{y \in \mathcal{Y}} p(y) H(X|Y = y)$
$= \sum_{y \in \mathcal{Y}} p(y) \sum_{x \in \mathcal{X}} p(x|y) \log_2(1/p(x|y))$

conditional entropy

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the chain rule

for any rvs X, Y:

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

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