Multiple fare-class capacity allocation

- n fare classes, capacity c
- prices $P_1 \leq P_2 \leq \ldots \leq P_n$

  (demand, ~distrib) $(D_1, \sim F_1)$ $(D_2, \sim F_2)$ $(D_n, \sim F_n)$

- Assumptions
  - i) D_i are independent
  - ii) D_i realized before D_{i+1} (sequential arrival)

- Controls - protection levels (nested)
  - $x_j \equiv$ protection level for fare classes $j+1, j+2, \ldots, j_n$

- Let $S_j \equiv$ Capacity available when fare class j arrives

  $(S_1 = c)$

then

$$V_j(S_j) = \max_{x_j \in \underbrace{\{0, 1, \ldots, S_j\}}_{A(S_j)}} \mathbb{E}_{F_j}\Big[\underbrace{P_j \min\{D_j, S_j - x_j\}}_{R_j(S_j, x_j, D_j)} + V_{j+1}(S_{j+1})\Big]$$

where $S_{j+1} = \max\{S_j - D_j, x_j\}$

- Question - Why protection levels?

- Alternative - Let's solve an 'easier' problem
  - Assumption 1 - We know D_j before allocating class j seats
  - Assumption 2 - Can choose exact number of seats for j

- $$\bar{V}_j(s_j) = \mathbb{E}_{F_j}\left[\max_{x \in \{0,1,\dots,s_j\}} \left\{ p_j \cdot \min(x, D_j) + \bar{V}_{j+1}(s_{j+1}) \right\}\right]$$

$$\max(s_j - x, s_j - D_j) = s_j - \min(x, D_j)$$

- Easier state variable - Let $\underbrace{y_j}_{\text{final desired state}} = s_j - \min(x, D_j)$

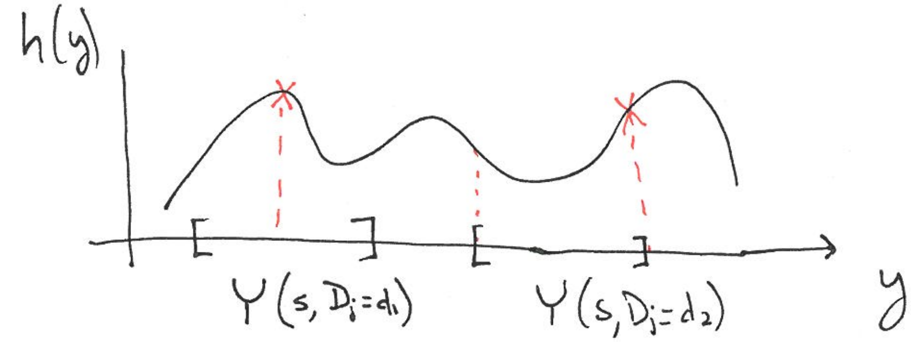
$$\bar{V}_j(s_j) = \mathbb{E}_{F_j}\left[\max_{y_j \in \underbrace{\{(s_j - D_j)^+, \dots, s_j\}}_{\text{random set } Y(s_j, D_j)}} p_j(s_j - y_j) + \bar{V}_{j+1}(y_j)\right]$$

$$\Rightarrow \boxed{\bar{V}_j(s_j) = p_j s_j + \mathbb{E}_{F_j}\left[\max_{y \in Y(s_j, D_j)} \underbrace{\left(-p_j y_j + \bar{V}_{j+1}(y_j)\right)}_{h_j(y_j)}\right]}$$

What can we say about the solution?

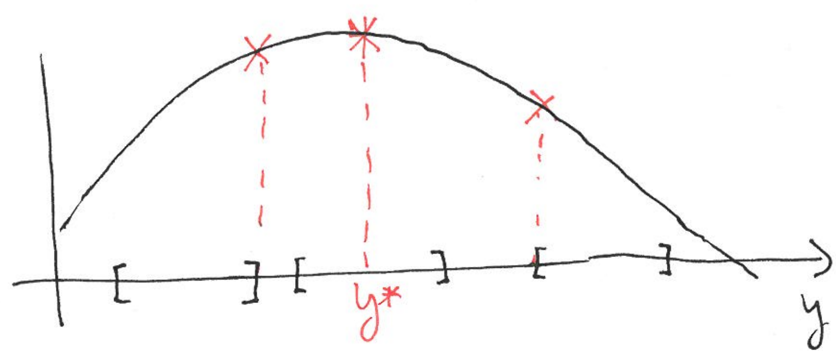
- Assumption 3 - Consider continuous D_j

- Maximizing a fixed fn over a random set (interval)



  - In general, this is some complicated random variable

- What if $h(y)$ is concave



Let interval be $[L, U]$

soln $\equiv$

$$y^*([L,U]) = \begin{cases} L ; & L > y^* \\ y^* ; & L \le y^* \le U \\ U ; & y^* > U \end{cases}$$

(Do we need concave? What about  ?)

see assignment

(Refer : Quasiconcavity) (What if $y \in$ discrete set ?)

- In our case - $\mathbb{E}_{F_j}\left[\max_{y \in \{(s_j - D_j)^+, \dots, s_j\}} -P_j y_j + \bar{V}_{j+1}(y_j)\right]$

(Assn 4) Suppose $\bar{V}_{j+1}(\ )$ is concave $\Rightarrow h_j(y_j) = -P_j y_j + \bar{V}_{j+1}(y_j)$ is concave in y_j

- Define $\ = \arg\max_{y \in [0,c]} \{h_j(y)\} = x_j^*$

Then opt $y_j^* = \begin{cases} (s_j - D_j)^+ ; & (s_j - D_j)^+ > x_j^* \\ x_j^* ; & s_j - D_j \le x_j^* \le s_j \\ s_j ; & s_j < x_j^* \end{cases}$

* Thus - given **A1** - 'Oracle access' to D_j

A2 - Control $\equiv$ Exact # of seats at fare p_j

A3 - Continuous D_j

A4 - Concave $\bar{V}_j(\cdot)$

then optimal capacity allocation $\Rightarrow$ protection levels $\{x_j^*\}$

- where $x_j^* = \underset{x \in [0,c]}{\arg\max}\ h_j(x) = \underset{x \in [0,c]}{\arg\max} \left(-p_j x + \bar{V}_{j+1}(x)\right)$

- we don't need to store the value fns! Only x_j^*

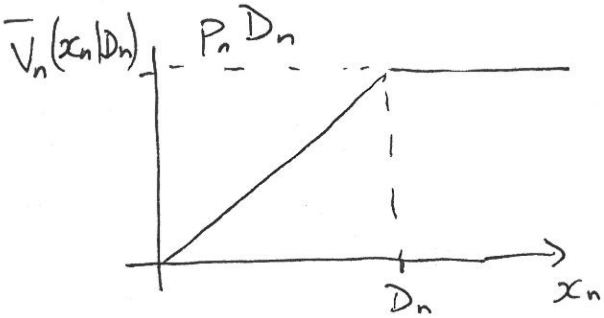
* Let's remove the assumptions - A4, then A1 and A2

* **Concavity of $\bar{V}_j(\,)$**

- Induction on $n, n-1, \ldots, 1$

- $$\bar{V}_n(x_n \mid D_n) \triangleq \max_{(x_n - D_n)^+ \le y \le x_n} \left[-p_n (x_n - D_n)^+ + p_n x_n\right]$$

(or $p_n \cdot \min\{x_n, D_n\}$



- concave in x $\forall D_n$

- $\bar{V}_n(x_n) = \mathbb{E}_{F_n}\left[\bar{V}_n(x_n \mid D_n)\right]$

$=$ Concave in x_n

(linear combination of concave fns)

- Assume $\bar{V}_{j+1}(x_{j+1})$ is concave in x_{j+1}

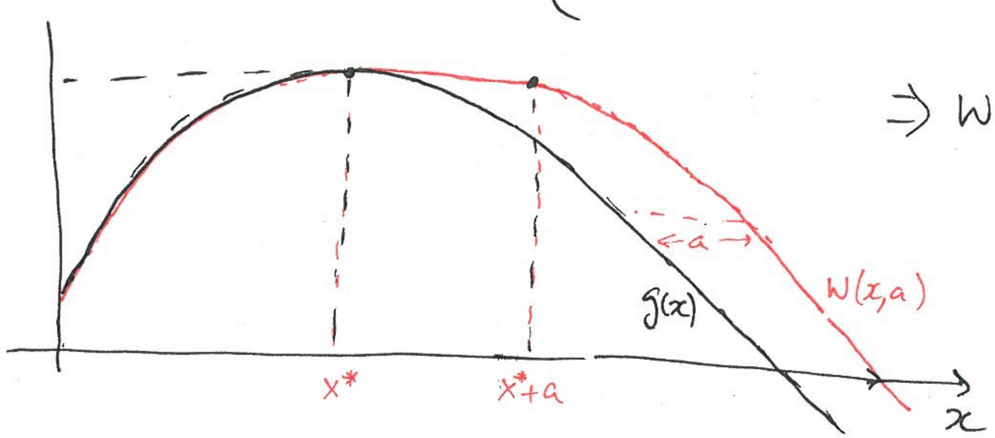
Let $F_j(y) = \left[-p_j\, y + \bar{V}_{j+1}(y)\right]$

$$V_j(x_j \mid D_j) = \underbrace{\max_{(x_j - D_j)^+ \le y \le x_j} \left[F_j(y) \right] + p_j\, x_j}_{\text{sufficient to show this is concave}}$$

- Consider $W(x, a) = \max\limits_{(x-a) \le y \le x} \big(g(x) \big)$ ← concave

  - Let $x^* \equiv \arg\max g(x)$

  - then $y^*(x) = \begin{cases} x \; ; & x \le x^* \\ x^* \; ; & x - a \le x^* \le x \\ x - a \; ; & x^* \le x - a \end{cases}$



$\Rightarrow W(x,a) = g(y^*(x))$ is concave $\forall\ a$

- Notes - this works for discrete demands (~~A3~~)
  - Protection levels are optimal even if we control exact allocations (~~A2~~)
  - What about A1? Claim - It does not matter...