ORIE 6180: Design of Online Marketplaces **Spring 2016**

Lecture 5 — February 17

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We actually used Tonelli's Theorem in this class.

5.1 Overview of the last lecture

Last lecture we proved Myerson's Lemma:

Lemma 5.1. *For single parameter settings an allocation rule $x(b)$ is implementable if and only if it is monotone. Moreover, there is a unique payment rule (assuming $p_i(0) = 0$)*

$$p_i(v_i) = v_i x_i(v_i, v_{-i}) - \int_0^{v_i} x_i(z, v_i)dz = \int_0^{v_i} x_i'(z, v_{-i})dz$$

Recall that an allocation rule $x(b)$ is monotone if $\forall i, b_{-i}$, $x_i(z, b_{-i})$ is non-decreasing in z and implementable if there exists a payment-rule p, such that (x, p) is DSIC.

5.2 Overview of this lecture

In this lecture we use Myerson's lemma to design a DSIC maximizing the revenue $R = \sum_i p_i(b)$. The extension thereof to BIC mechanisms can be found in [1], Chapter 1 and 2.

5.3 Maximizing Revenue

Consider n agents with values $v_i \sim F_i$, $F_i \perp\!\!\!\perp$. We aim to design a DSIC mechanism to maximize $\mathbb{E}[R] = \mathbb{E}_{\{F_i\}}[\sum_i p_i(v_i, v_{-i})]$.

By Myerson's Lemma, there is a unique one payment rule for which an allocation rule $x(b)$ is DSIC.

We begin by expressing $\mathbb{E}[R]$ in terms of the allocation $x(b)$:

$$\mathbb{E}[R] = \mathbb{E}_{F_1,\dots,F_n}[\sum_i p_i(v_i, v_{-i})] = \sum_i \mathbb{E}_{F_1,\dots,F_n}[p_i(v_i, v_{-i})].$$

Notice that the second inequality holds by linearity of expectation. We now assume that $v_i \in [0, v_{\max}]$ and $\frac{dF_i(z)}{dz} = f_i(z)$. We can then write

$$\mathbb{E}_{F_i,F_{-i}}[p_i(v_i, v_{-i})]$$

$$= \mathbb{E}_{F_i}\Big[\int_0^{v_i} x_i'(z, v_{-i}) z dz\Big]$$

$$= \int_0^{v_{\max}} \Big[\int_0^{v_i} x_i'(z, v_{-i}) z dz\Big] f_i(v_i) dv_i$$

$$= \int_0^{v_{\max}} \Big[\int_z^{v_{\max}} f_i(v_i) dv_i\Big] z x'(z, v_{-i}) dz$$

$$= \int_0^{v_{\max}} \Big[1 - F_i(z)\Big] z x'(z, v_{-i}) dz$$

$$= \Big[(1 - F_i(z) z x_i(z, v_{-i})\Big]_0^{v_{\max}} - \int_0^{v_{\max}} \Big[1 - F_i(z) - z f_i(z)\Big] x_i(z, v_{-i}) dz$$

$$= \int_0^{v_{\max}} \Big(z - \frac{1 - F_i(z)}{f_i(z)}\Big) x_i(z, v_{-i}) f_i(z) dz$$

$$= \mathbb{E}_{F_i}\Big[\phi_i(v_i) x_i(v_i, v_{-i})\Big],$$

where $\phi(z) = z - \frac{1-F_i(z)}{f_i(z)}$ is the virtual valuation function of i.

Notice that the first equality above holds by independence and the payment rule in Myerson's Lemma. The third equality holds by Tonelli's Theorem as x is monotone, implying that the x_i are non-decreasing and $x_i' \geq 0$. The fifth equality is integration by parts.

We can then express

$$\mathbb{E}[R] = \sum_i \mathbb{E}_{F_i,F_{-i}}[\phi_i(v_i) x_i(v_i, v_{-i})] = \mathbb{E}_F\Big[\sum_i \phi(v_i) x_i(v_i, v_{-i})\Big],$$

implying that maximizing revenue is the same as maximizing *virtual welfare*.

5.3.1 Example: single item, single bidder

We maximize $\mathbb{E}[R]$:

$$\mathbb{E}[R] = \mathbb{E}[\phi(v) x(v)] = \mathbb{E}\Big[\Big(v - \frac{1 - F(v)}{f(v)}\Big) x(v)\Big].$$

It is easy to see that this is maximized if we set $x_v = 1$ if and only if $v - \frac{1-F(v)}{f(v)} \geq 0$. We then charge $p = \phi^{-1}(0)$, which is the same as the monopoly price that maximized $p(1 - F(p))$ as

$$\frac{d}{dp} p(1 - F(p)) = \Big(\frac{1 - F(p)}{f(p)} - p\Big) f(p)$$

is 0 for $p = \frac{1-F(p)}{f(p)}$ and for such p we have $\phi(p) = 0$.

5.3.2 Example: n agents, $F_i \sim F$ (i.i.d.)

We now have $\mathbb{E}[R] = \mathbb{E}_{F^N}[\sum_i \phi(v_i)x(v_i, v_{-i})]$.

To maximize the virtual surplus, we solicit bids b_i and simulate VCG on ϕ, i.e.: $(x, p) = VCG(\phi(b))$.

Definition 1. *$VCG(\phi(b))$ is monotone if and only if $\phi(\cdot)$ is monotone (non-decreasing). We call distributions giving rise to non-decreasing ϕ* regular distributions.

Remark: Notice that ϕ is monotone for uniform, exponential and most other natural distributions. It is easy though to construct examples here ϕ is not monotone using mixed distribution.[1]

Consider the auction of a single item, n agents with i.i.d. values coming from a regular distribution. Let $(x, p) = VCG(\phi(b))$. Define the critical value for bidder is as the value v_i^+, when i starts to get i, i.e.: $\phi(v_i^+) = \max_{j \neq i}\{\phi(v_j), 0\}$. Then the resulting price for i is

$$p_i = \max\left\{\max_{j \neq i}\{\phi^{-1}(\phi(v_j))\}, \phi^{-1}(0)\right\} = \max\left\{\max_{j \neq i}\{v_j\}, \phi^{-1}(0)\right\}.$$

[1]If ϕ is not regular, apply a technique called ironing.

Bibliography

[1] Hartline, J. D. Mechanism design and approximation.