ORIE 6180: Design of Online Marketplaces **Spring 2016**

Lecture 7 — February 24

Lecturer: Sid Banerjee *Scribe: M. Sumida and Z. Tan*

7.1 Overview of the last lecture

Simple near optimal auctions for single item (with independent valuation distributions among bidders) based on Prophet Inequality.

7.2 Overview of this lecture

Definition of Bayes-Nash equilibrium, discussion about VCG mechanisms and Clarke pivot rule, introduction of Myerson Satterthwaite theorem.

7.3 The threshold policy (Review of last time)

Calculate threshold virtual value $\hat{\lambda}$; Ask bids, then keep all agents s.t. $\phi_i(v_i) \geq \hat{\lambda}$; Result in k agents $i_1, ..., i_k$; If agent i is awarded the item, then she pays $p_i = \phi_i^{-1}(\hat{\lambda})$; Then given this payment, find the allocation rule: any allocation rule that can ensure DSIC works, e.g.,1)based on priority order; 2)based on lottery.

7.4 Recall the Revelation Principle

Given a mechanism and an equilibrium concept, there exit a sealed bid direct revelation mechanism which is incentive compatible and implements the same allocation and payment rule.

7.5 Bayes-Nash equilibrium

7.5.1 Definition

In Bayes-Nash Equilibrium, there are n agents, each agent i has type θ_i, F is the joint distribution $\{\theta_i\}^n$, each agent i chooses a strategy $s_i : \theta_i \to A$, where A is the action space. Let vector $\mathbf{s} = (s_1, ..., s_n)$. Bayes-Nash Equilibrium for a game G with type distribution F is a strategy profile $\mathbf{s}$ s.t. for every agent i, $s_i(\theta_i)$ is the best response she can do when other players play $\mathbf{s}_{-i}(\theta_{-i})$, where $\theta_{-i} \sim F_{\theta_{-i}}(\cdot|\theta_i)$.

7.5.2 3 stages of knowledge

ExAnte: every agent i knows F;
Interim: every agent i knows θ_i and $F_{\theta_{-i}}(\cdot|\theta_i)$;
Expost: every agent i knows θ_i and $F_{\theta_{-i}}(\cdot|\theta_i)$, plus all actions $a_i, i = 1, ...n$.

7.6 VCG mechanisms and Clarke Pivot Rule

7.6.1 VCG mechanisms

Assume independent valuation distributions F_i, the social welfare is

$$W(\mathbf{v}) = \sum_i x_i(v_i)v_i, \tag{7.1}$$

and in the single item case (since $x_i = 0, 1$),

$$W(\mathbf{v}) = \sum_i x_i(v_i)v_i. \tag{7.2}$$

The welfare under VCG mechanism as

$$W^{VCG}(\mathbf{v}) = \max_{\mathbf{x}\in\chi} \sum_i v_i(\mathbf{x}), \tag{7.3}$$

i.e., it chooses the social welfare maximizing allocation rule

$$\mathbf{x}^{VCG} = \max_{\mathbf{x}\in\chi} W(\mathbf{v}). \tag{7.4}$$

And define everyone else's valuation under VCG mechanism as

$$W_{-i}^{VCG}(\mathbf{v}_{-i}) = \sum_{j\neq i} v_j(\mathbf{x}^{VCG}). \tag{7.5}$$

It sets the payments as

$$p_i^{VCG}(\mathbf{v}) = -\sum_{j\neq i} v_j(\mathbf{x}^{VCG}) + h_i(\mathbf{v}_{-i}). \tag{7.6}$$

So the utility of agent i with bid v_i' is

$$\begin{aligned} u_i^{VCG}(v_i') &= v_i(\mathbf{x}^{VCG}(v_i', \mathbf{v}_{-i})) - p_i^{VCG}(\mathbf{v}) \\ &= v_i(\mathbf{x}^{VCG}(v_i', \mathbf{v}_{-i})) + \sum_{j\neq i} v_j(\mathbf{x}^{VCG}) - h_i(\mathbf{v}_{-i}) \\ &= W^{VCG}(v_i', \mathbf{v}_{-i}) - h_i(\mathbf{v}_{-i}), \qquad (7.7) \end{aligned}$$

where h_i is a special function of i^{th} bidder that only depends on others' bids.
So under VCG, bidder i chooses the bid in the following way

$$\max_{v_i'} u_i(v_i') = \max_{v_i'} \left[W^{VCG}(v_i', \mathbf{v}_{-i}) - h_i(\mathbf{v}_{-i})\right]. \tag{7.8}$$

We call function h_i the "pivot rule", designing a proper pivot rule is the key for achieve simultaneously the following conditions:

1. Welfare maximization
2. Incentive Compatibility (IC)
3. Individual Rationality (IR)
4. No payment to agents by the seller

We will introduce one important pivot rule that satisfies $1 \sim 4$.

7.6.2 Clarke Pivot Rule

Note that by design of the VCG mechanism itself, 1 and 2 above are free no matter how we select $h_i(v_{-i})$. Welfare maximization is got for free since by the way of choosing $\mathbf{x}$ in VCG (8.3), and IC is also automatically satisfied because p_i is independent of the bid of agent i (8.6). So our goal is to design a h_i that make it satisfy the last two conditions, i.e., nonnegative utility for each agent and no payment to agents by the seller. For example, if we have simply $h_i = 0$, then by (8.6), p_i can be negative, i.e., the seller needs to pay the bidders, which is not desirable. So we want a better pivot rule.

The so called *Clarke's Pivot Rule* is:

$$h_i(\mathbf{v}_{-i}) = \max_{\mathbf{x}\in\chi} \sum_{j\neq i} v_j(\mathbf{x}) \tag{7.9}$$

Assume $v_i \in [v_{min}, v_{max}]$ where possibly $v_{min} < 0$. We could interpret a negative v_i as the seller's cost. Then we can rewrite bidder i's payment rule as:

$$\begin{aligned} p_i^{VCG}(\mathbf{v}) &= -\sum_{j\neq i} v_j(\mathbf{x}^{VCG}) + \max_{\mathbf{x}\in\chi} \sum_{j\neq i} v_j(\mathbf{x}) \\ &= -W_{-i}^{VCG}(\mathbf{v}_{-i}) + W^{VCG}(v_{min}, \mathbf{v}_{-i}) \end{aligned} \tag{7.10}$$

The Clarke Pivot Rule has the following properties:

Proposition 1. $u_i(v_{min}) = 0$

Proposition 2. *Among all mechanisms that are IC, IR, and welfare maximizing, VCG has the highest expected utilities for each bidder*

Proof: Let A be any other auction that is IC, IR and welfare maximizing.

$$\mathbb{E}_{-i}[u_i^A(v_i) - u_i^{VCG}(v_i)] = c_i$$

where c_i is some constant independent of v_i. This is because the only difference in the two utilities will be h_i term which does not depend on bidder i's bid.

Suppose FTSOC this constant is negative, i.e.

$$\mathbb{E}_{-i}[u_i^A(v_i) - u_i^{VCG}(v_i)] < 0$$

Consider the case $v_i = v_{min}$. We know $u_i^{VCG}(v_{min}) = 0$,

$$\implies \mathbb{E}_{-i}[u_i^A(v_{min})] < 0$$

$\implies$ A is not IR $\Rightarrow\Leftarrow$ $\square$

7.6.3 Example: Bilateral Trade

Consider the case where there is one buyer with value $v = v_{buyer} \in [v_{min}, v_{max}]$ and one seller with cose $c \in [c_{min}, c_{max}]$. Can think of the seller's value as $v_{seller} = -c \in [-c_{max}, -c_{min}]$

The possible allocations are $\chi = \{\text{sale}, \text{no sale}\}$. VCG auction will allocate *sale* if $v > c$ and *no sale* otherwise.

If a sale occurs, the VCG payments will be:

$$p_{seller} = W^{VCG}(-c_{max}, v) - v_{buyer}^{VCG}(-c, v) = \begin{cases} v - c_{max} - v = -c_{max} & \text{if } v > c_{max} \\ 0 - v = -v & \text{otherwise} \end{cases}$$

and

$$p_{buyer} = W^{VCG}(v_{min}, -c) - v_{seller}^{VCG}(-c, v) = \begin{cases} v_{min} - c + c = v_{min} & \text{if } v_{min} > c \\ 0 + c = c & \text{otherwise} \end{cases}$$

We can rewrite the payments as:

$$p_{seller} = -\min v, c_{max} \qquad\qquad p_{buyer} = \max(v_{min}, c)$$

Claim: If $v_{min} < c_{max}$ and $c_{min} < v_{max}$, then $p_{buyer}^{VCG} + p_{seller}^{VCG} < 0$ The above implies that the mechanism must make a payment to the seller.

Theorem 7.1 (Myerson-Satterthwaite). *In the bilateral trade setting, no mechanism can satisfy all 4 of the following properties:*

1. *Welfare maximizing*
2. *IR*
3. *IC*
4. *Budget balancing*