The Parisi and Coppersmith-Sorkin 'Conjectures'

- **Setting** – $m \times n$ matrix A of iid non-negative r.v.s
  – Want to pick k entries which form a matching (i.e., at most one entry per row/column)
  – $C_{k,m,n} \equiv$ weight of minimum k-matching
  ($C_n \equiv$ wt of min matching when $m=n$)

- **Results** – If $A_{ij} \sim U[0,1]$, then $\mathbb{E}[C_n]$ bounded as $n \to \infty$ (Walkup '79)
  – If $A_{ij} \sim Exp(1)$, then $\mathbb{E}[C_n] \longrightarrow \pi^2/6$ as $n \to \infty$
  (conjecture by Parisi & Mezard '87, proved by Aldous '00)

  – (The Parisi Conjecture) If $A_{ij} \sim Exp(1)$, then

$$\mathbb{E}[C_n] = \sum_{i=1}^{n} 1/i^2$$

  – (The Coppersmith-Sorkin Conjecture) If $A_{ij} \sim Exp(1)$, then

$$\mathbb{E}[C_{k,m,n}] = \sum_{\substack{i,j \geq 0 \\ i+j<k}} \frac{1}{(m-i)(n-j)}$$

(conjecture by Parisi '98, C-S '99; proved by Linusson-Wästlund and Nair-Prabhakar-Sharma '03)

  – Moreover, $C_n \longrightarrow \pi^2/6$ in probability (ie, $Var(C_n) \searrow 0$)

- We will see an argument by J. Wästlund from "An Easy Proof of the $\zeta(2)$ Limit in the Random Assignment Problem" (2009)

Background and some heuristic calculations

- If $X \sim \mathcal{E}xp(\lambda)$ ('rate'), then we have the following
  i) $f(x) = \lambda e^{-\lambda x}$, $F(x) = 1 - e^{-\lambda x}$ $\forall x \geq 0$
  ii) $\mathbb{E}[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$
  iii) $\mathbb{P}[X \leq s+t \mid X \geq s] = \mathbb{P}[X \leq t]$ (memoryless)
- Given $X_1, X_2, \ldots, X_n$ independent, with $X_i \sim \mathcal{E}xp(\lambda_i)$
  $\Rightarrow$ $\min_{i\in[n]}\{X_i\} \sim \mathcal{E}xp(\sum_{i=1}^n \lambda_i)$, $\arg\min_{i\in[n]}\{X_i\} \sim i$ wp $\frac{\lambda_i}{\sum_{i=1}^n \lambda_i}$

Returning to $A \sim m\times n$ matrix of iid $\mathcal{E}xp(1)$ r.v.s

- $\min_{(i,j)\in[m]\times[n]}\{A_{ij}\} \sim \mathcal{E}xp(mn)$ (and hence $\mathbb{E}[C_{1,m,n}] = 1/mn$)

- For a lower bound, consider picking the k lowest entries

$$\Rightarrow \mathbb{E}[C_{k,m,n}] \geq \frac{1}{mn} + \underbrace{\left(\frac{1}{mn} + \frac{1}{mn-1}\right)}_{\text{by memorylessness, } \mathbb{E}[\text{second smallest } A_{ij} \mid \text{smallest} = s] = s + 1/mn-1} + \ldots + \left(\frac{1}{mn} + \frac{1}{mn-1} + \ldots + \frac{1}{mn-k+1}\right)$$

For $m=n=k$, we get $\mathbb{E}[C_{n,n,n}] \geq \sum_{i=0}^{n-1} \frac{n-i}{n^2-i} \in \left[\frac{1}{n^2}, \frac{1}{n^2-n}\right]\cdot\left(\sum_{i=1}^{n} i\right)$

$$\in \left[\frac{1}{2} + \frac{1}{2n}, \frac{1}{2} + \frac{1}{n-1}\right] \xrightarrow{n\nearrow\infty} \frac{1}{2}$$

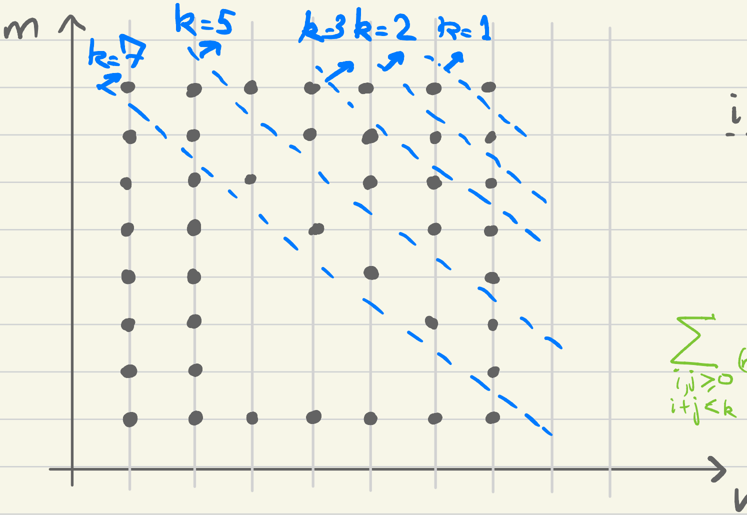
- For an upper bound, consider a greedy policy

$$\Rightarrow \mathbb{E}[C_{k,m,n}] \leq \frac{1}{mn} + \underbrace{\left(\frac{1}{mn} + \frac{1}{(m-1)(n-1)}\right)}_{\text{We delete the minimum edge, remove the row and column, and find min in remaining } (n-1)(n-1) \text{ graph}} + \ldots + \left(\frac{1}{mn} + \frac{1}{(m-1)(n-1)} + \ldots + \frac{1}{(m-k+1)(n-k+1)}\right)$$

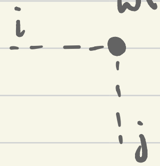
Thus $\mathbb{E}[C_{n,n,n}] \leq \sum_{i=0}^{n-1} \frac{n-i}{(n-i)^2} = H_n \in [\ln(n+1), \ln(n)+1]$
($\nearrow \infty$ as $n \nearrow \infty$)

- Thus, the greedy policy here is very poor! This is in contrast to max wt matching, where greedy is a 2-approximation.
  - In fact, getting any constant upper bound here is somewhat non-trivial.

A graphical view of the C-S formula



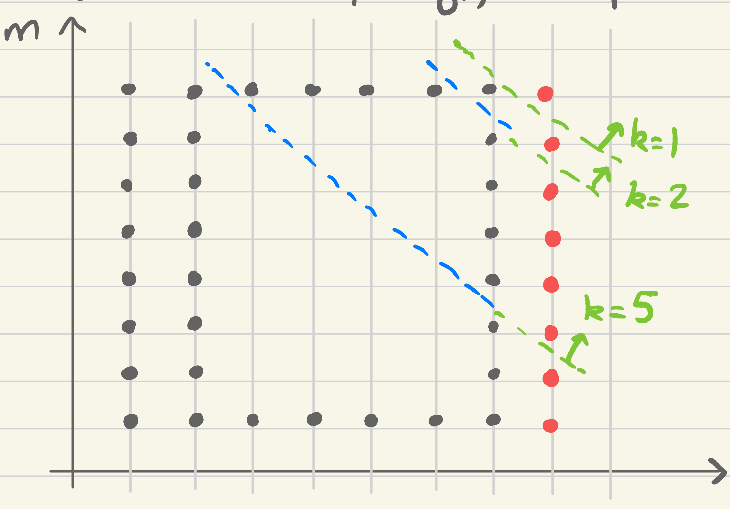
$m = 8$
$n = 7$



$wt = 1/i \cdot j$

$$\sum_{\substack{i,j \geq 0 \\ i+j<k}} \frac{1}{(m-i)(n-j)} = \sum_{\substack{i,j \geq 1 \\ i+j \geq m+n-k}} 1/i \cdot j$$

This leads to a (surprising!) decomposition for induction



$m = 8$
$n = 8$

$$\boxed{\mathbb{E}\left[C_{k,m,n}\right] = \mathbb{E}\left[C_{k-1,m,n-1}\right] + \frac{1}{n} \sum_{i=1}^{m} \frac{1}{i}}$$

The Participation Lemma (Buck-Chan-Robbins)

Given $A \equiv m \times n$ with $A_{ij} \sim \text{Exp}(1)$ iid

let $\hat{A} \triangleq \left[\begin{array}{c} A \\ \hdashline B \end{array}\right]$, where $B = 1 \times n$, $B_j \sim \text{Exp}(\lambda)$ iid

some parameter

- Since entries are from a continuous r.v $\Rightarrow \exists$ unique min matchings (ie, matrix $A, \hat{A}$ are 'generic')

Denote $M_k^* \equiv$ unique min k-matching of $\hat{A}$

Lemma - Let $\hat{A}$ be an $m \times n$ matrix, $k < \min(m,n)$ and let M_k^* denote the min-wt k-matching. Then every row which participates in M_k^* also participates in M_{k+1}^*

Pf - Augmenting paths!

In particular, argue that the edges in the symmetric difference of M_k^*, M_{k+1}^* form a single alternating path (no cycles)

Lemma - For any $k \leq \min(m,n)$

$$\boxed{\mathbb{P}[B \text{ participates in } M_{k+1}^* \mid B \text{ does not participate in } M_k^*] = \frac{\lambda}{\lambda + m - k}}$$

Pf - Suppose wlog the first k rows participate in M_k^*

- By previous lemma, one more row in $\{A_{k+1}, A_{k+2}, \dots, A_m, B\}$ ends up participating in M_{k+1}^*. Assume the new element is in column j

- $\mathbb{P}[B \in M_{k+1}^* \mid B \notin M_k^*] = \mathbb{P}[\underbrace{B_j}_{\text{Exp}(\lambda)} < \underbrace{A_{i,j} \ \forall i \in \{k+1, \dots, m\}}_{\min_i (A_{ij}) \sim \text{Exp}(m-k)}]$

$$= \frac{\lambda}{\lambda + m - k}$$

Proof of the C-S formula

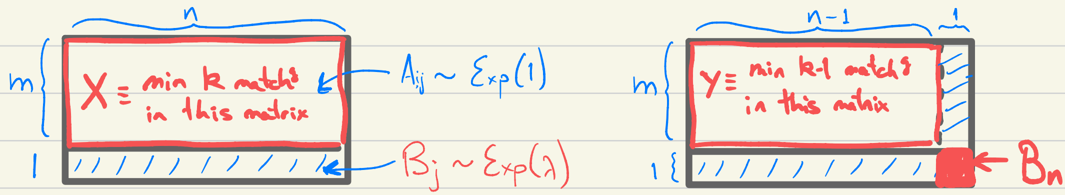
- We want to show by induction that

denote this as Δ_{kmn}

$$\boxed{\mathbb{E}[C_{k,m,n}] - \mathbb{E}[C_{k-1,m,n-1}] = \sum_{i=0}^{k-1} \frac{1}{n(m-i)}} \quad (*)$$

- Recall $\hat{A}$ = A augmented with an extra row B with
$A_{ij} \sim Exp(1)$, $B_j \sim Exp(\lambda)$ iid

Let X = wt of min k-matching in $A = \{A_{ij}\}_{i\le m, j\le n}$
Y = wt of min $(k-1)$-matching in $\{A_{ij}\}_{i\le m, j\le n-1}$



$$\boxed{\textbf{Claim 1} - \quad X \sim C_{k,m,n} \quad , \quad Y \sim C_{k-1,m,n-1}}$$

($\sim$ = 'distributed as')

- This is by definition of C_{kmn} and X, Y
- As a result, $\mathbb{E}[\Delta_{kmn}] = \mathbb{E}[X - Y]$

Also, iterating over the participation lemma, we get

$$\boxed{\textbf{Claim 2} - \quad \mathbb{P}[B \in M_k^*] = 1 - \prod_{i=0}^{k-1}\left(\frac{m-i}{m-i+\lambda}\right) = \lambda\left(\sum_{i=0}^{k-1}\frac{1}{m-i}\right) + O(\lambda^2)}$$

Next, by symmetry, we have

$$\mathbb{P}[\underset{\text{edge}}{B_n} \in M_k^*] = \frac{1}{n}\mathbb{P}[\underset{\substack{\uparrow \\ \text{row/node}}}{B} \in M_k^*] = \frac{\lambda}{n}\sum_{i=0}^{k-1}\frac{1}{m-i} + O(\lambda^2)$$

Finally, we want to express $\mathbb{P}[B_j \in M_k^*]$ in terms of X, Y

$$\boxed{\text{Claim 3} - \quad \mathbb{P}[B_n \in M_k^*] = \mathbb{P}[B_n < X - Y] - O(\lambda^2)}$$

Warning - the next bit is somewhat sketchy (though true ...)

To see why we need the additional $O(\lambda^2)$ term, note that if $B_n < X - Y$, then the only way $B_n \notin M_k^*$ is if some other B_j is part of an even smaller k matching. For this to happen, we must have $\min_{j \neq n} B_j < \underbrace{X}_{\text{min match}^g \text{ without } B}$

However, given n iid $\mathrm{Exp}(\lambda)$ rv, the probability that two of them are less than X is at most (by union bound)

$$\mathbb{E}\left[\binom{n}{2}\left(1-e^{-\lambda X}\right)^2\right] \le \mathbb{E}[n^2\lambda^2X^2] \le n^2\lambda^2\,\mathbb{E}[X^2] = O(\lambda^2)$$

(note - here n is fixed, so $\mathbb{E}[X^2] \le n^2$ which is the sum of all variances.)

Putting things together

$$\mathbb{P}[B_n < X - Y] = \frac{\lambda}{n}\sum_{i=0}^{k-1}\frac{1}{m-i} + O(\lambda^2)$$

Also $\mathbb{P}[B_n < X - Y] = \mathbb{P}[B_n < \Delta_{k,m,n}] = 1 - \mathbb{E}[e^{-\lambda\Delta_{k,m,n}}]$

Thus $\mathbb{E}[\Delta_{k,m,n}] = \lim_{\lambda \to 0}\left[\frac{d}{d\lambda}\mathbb{P}[B_n < X - Y]\right] = \frac{1}{n}\sum_{i=0}^{k-1}\frac{1}{(m-i)}$