The Poisson Process

- (Continuous-time random process) Collection of indexed r.v. $X = (X_t \,;\, t \in \mathbb{R})$ on the same $(\Omega, \mathcal{F}, \mathbb{P})$. Can be viewed in 3 ways:
  - i) For fixed $t \in \mathbb{R}$, X_t is a fn on Ω (Slice view)
  - ii) X is a fn on $\mathbb{R} \times \Omega$, with value $X_t(\omega)$ for $(t, \omega) \in \mathbb{R} \times \Omega$ (Joint view)
  - iii) For fixed $\omega \in \Omega$, $X_t(\omega)$ is a fn of t (sample-path view)
- (Counting function) fn $f : \mathbb{R}_+ \to \mathbb{N}$ s.t. $f(0) = 0$, f is non-decreasing, right continuous
  - $f(t) = \#$ of 'counts' in $(0, t]$, $f(b) - f(a) = \#$ of counts in $(a, b]$
  - If t_i denotes the time of the i^{th} count, then $f \equiv (t_1, t_2, \ldots)$
    $t_i - t_{i-1} \equiv$ inter-arrival or inter count time
    $f(t) = \sum_{n=1}^{\infty} \mathbb{1}\{t \geq t_n\}$, $t_n = \min\{t : f(t) \geq n\}$.
- A random process is called a counting process if its sample paths are counting fns w.p. 1

- Defn (Poisson process) A counting process N on $\mathbb{R}_+$ is called a homogeneous Poisson process (PP) with intensity $\lambda > 0$ if $\forall k \in \mathbb{N}$, and $\forall$ indices $0 \leq t_1 \leq t_2 \leq \ldots \leq t_k$ the process satisfies
  - i) $N(t_i) - N(t_{i-1}) \perp\!\!\!\perp N(t_j) - N(t_{j-1}) \;\; \forall i \neq j,\; i, j \in [k]$
  - ii) $N(t_i) - N(t_{i-1}) \sim \mathrm{Poi}(\lambda(t_i - t_{i-1})) \;\; \forall i \in [k]$

- Does such a process exist? And why do we want Poisson increments?

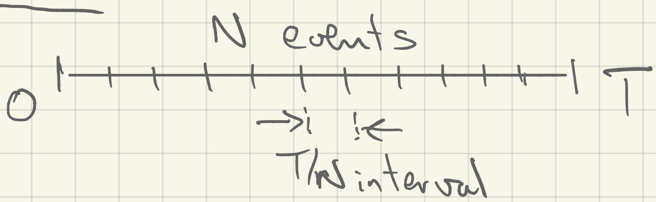
- Aside - Suppose we want a discrete-time counting process which is time-homogeneous + Markovian - in other words we want $N = (N(n); n \in \mathbb{N})$ s.t $\mathbb{P}[N(n+1) = k \mid N(n) = j] = P_{jk} \ \forall n$, and $\mathbb{P}[N(n) = k \mid N(n-1), N(n-2), \ldots, N(1)] = \mathbb{P}[N(n) = k \mid N(n-1)]$

  - Simplest candidate: Sum of Bernoulli trials, ie,
    $X(i) \sim Ber(p)$, iid, $N(n) = \sum_{i=1}^{n} X(i)$, $N(0) = 0$
  - Inter-count times $T_i - T_{i-1} \sim Geom(p)$
  - For any $0 \leq n_1 \leq n_2 \leq \ldots \leq n_k$, we have
    - $N(n_i) - N(n_{i-1}) \perp\!\!\!\perp N(n_j) - N(n_{j-1}) \ \forall i \neq j$
    - $N(n_i) - N(n_{i-1}) \sim Bin(n_i - n_{i-1}, p)$
  - Moreover, more complex HMCs can be 'built up' using this basic process!

How can we go from here to create a continuous time counting process?

Idea 1 - Replace $T_i - T_{i-1}$ with a continuous distribution. In particular, to preserve Markovian nature, we would want the distribution to be memoryless $\Rightarrow T_i - T_{i-1} \sim Exp(\lambda)$

Idea 2 - 'Scale' the Bernoulli process to make it continuous time in the limit.



- Replace $\sum_{t=1}^{T} X_t$ with $\frac{1}{N} \sum_{t=1}^{NT} X_t$
- Thus $N(t) \sim Bin(Nt, P/N)$

What happens when $N \to \infty$?

• Amazingly both ideas give the same result - the Poisson process!

<u>Thm</u> (Characterizations of the Poisson Process) Let $N(t)$ be a counting process on $\mathbb{R}_+$, and $\lambda \geq 0$. The following are all equivalent

i) N is a Poisson process with rate λ, ie, $\forall k \in \mathbb{N}$ and $0 \leq t_1 \leq \dots \leq t_k$, we have

Poisson point process

- $N(t_i) - N(t_{i-1}) \perp\!\!\!\perp N(t_j) - N(t_{j-1}) \quad \forall\, i \neq j$
- $N(t_i) - N(t_{i-1}) \sim \mathrm{Poi}\left(\lambda(t_i - t_{i-1})\right)$

ii) Inter-count times $T_1, T_2 - T_1, T_3 - T_2, \dots$ are iid $\mathcal{E}xp(\lambda)$

Exponential inter-count times

ie. $\mathbb{P}[T_i - T_{i-1} \leq k] = 1 - e^{-\lambda k}$

iii) For any $t \geq 0$ and $\delta \geq 0$

Indep stationary increments

$$\mathbb{P}[N(t+\delta) - N(t) = 0] = 1 - \lambda\delta + o(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) = 1] = \lambda\delta + o(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) > 1] = o(\delta^2) \quad \text{('simple' process)}$$

(where $f(\delta) = o(\delta) \iff \lim_{\delta \to 0} f(\delta)/\delta = 0$)

iv) For any $t \geq 0$, $N(t) \sim \mathrm{Poi}(\lambda t)$, and given $\{N_t = n\}$, the count times $T_1, T_2, \dots, T_n$ are uniform in $[0,t]$, ie.

uniform arrivals in an interval

$$f(t_1, t_2, \dots, t_n \mid N_t = n) = \begin{cases} n!/t^n & ; 0 < t_1 < t_2 < \dots < t_n < 1 \\ 0 & ; \text{o.w.} \end{cases}$$

We now show some of these equivalences

(i) $\Rightarrow$ (iii) For any t, δ, we have $N(t+\delta) - N(t) \sim Poi(\delta\lambda)$

$$\Rightarrow \mathbb{P}[N(t+\delta) - N(t) = 0] = e^{-\lambda\delta} = 1 - \lambda\delta + O(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) = 1] = (\lambda\delta) e^{-\lambda\delta} = \lambda\delta + O(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) > 1] = 1 - e^{-\lambda\delta}(1+\lambda\delta)$$

$$= 1 - (1 - \lambda\delta + O(\delta^2)) - (\lambda\delta + O(\delta^2)) = O(\delta^2)$$

(ii) $\Rightarrow$ (iii) For any $t, \delta \geq 0$, Let $N(t) = k$

$$\mathbb{P}[N(t+\delta) - N(t) = 0] = \mathbb{P}[\underbrace{T_{k+1} - t}_{\sim \mathcal{E}xp(\lambda)} > \delta]$$

$$= e^{-\lambda\delta} = 1 - \lambda\delta + O(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) = 1] = \mathbb{P}[T_{k+1} - t < \delta, T_{k+2} > t+\delta]$$

$$= \int_0^\delta \lambda e^{-\lambda x} \cdot e^{-\lambda(\delta - x)} \, dx$$

$$= \lambda\delta e^{-\lambda\delta} = \lambda\delta + O(\delta^2)$$

$$\mathbb{P}[N(t+\delta) - N(t) > 1] = 1 - e^{-\lambda\delta} - \lambda\delta e^{-\lambda\delta} = O(\delta^2)$$

(i) $\Rightarrow$ (ii) Consider $T_k = t$, and T_{k+1}. Then

$$\{T_{k+1} - T_k > t\} \equiv \{A(t) = 0\}$$

$$\Rightarrow \mathbb{P}[T_{k+1} - T_k > t] = e^{-\lambda t} = 1 - F(t) \text{ for } F \sim \mathcal{E}xp(\lambda)$$

$$\Rightarrow T_{k+1} - T_k \sim \mathcal{E}xp(\lambda)$$

(iii) $\Rightarrow$ (i) For this, we prove a more general result on convergence of Binomials to Poisson rv

Thm (Poisson Approximation) Let $X_i \sim \mathrm{Ber}(p_i)$, independent, and $S_n = \sum_{i=1}^{n} X_i$. Moreover, let $P_n \sim \mathrm{Poi}(\lambda_n)$, where $\lambda_n = \sum_{i=1}^{n} p_i$. Then

$$d_{TV}(S_n, P_n) \leq 2 \sum_{i=1}^{n} p_i^2$$

Pf - Consider coupling $(X_i, Y_i) \in \{0,1\} \times \mathbb{N}_0$ s.t.

$$\mathbb{P}[X_i = x, Y_i = y] = \begin{cases} 1 - p_i & ; \; x = y = 0 \\ e^{-p_i} - (1 - p_i) & ; \; x = 1, y = 0 \\ p_i^y e^{-p_i}/y! & ; \; x = 1, y \geq 1 \end{cases}$$

Check that $X_i \sim \mathrm{Ber}(p_i)$, $Y_i \sim \mathrm{Poi}(p_i)$. Now set $S_n = \sum_{i=1}^{n} X_i$, $P_n = \sum_{i=1}^{n} Y_i$. Check that $P_n \sim \mathrm{Poi}(\lambda_n)$. Moreover

$$|\mathbb{P}[S_n = k] - \mathbb{P}[P_n = k]| = |\mathbb{P}[S_n = k, P_n \neq k] - \mathbb{P}[S_n \neq k, P_n = k]|$$

$$\leq \mathbb{P}[S_n = k, S_n \neq P_n] + \mathbb{P}[P_n = k, S_n \neq P_n]$$

$$\Rightarrow d_{TV}(S_n, P_n) = \sum_k |\mathbb{P}[S_n = k] - \mathbb{P}[P_n = k]| \leq 2\mathbb{P}[S_n \neq P_n]$$

Finally $\mathbb{P}[S_n \neq P_n] \leq \sum_{i=1}^{n} \mathbb{P}[X_i \neq Y_i]$

$$= \sum_{i=1}^{n} \left(e^{-p_i} - 1 + p_i + \mathbb{P}[Y_i \geq 2]\right)$$

$$= \sum_{i=1}^{n} p_i(1 - e^{-p_i}) \leq \sum_{i=1}^{n} p_i^2 \qquad \blacksquare$$

Thus if $X_i \sim \mathrm{Ber}(\lambda/N)$ and $S_N = \sum_{i=1}^{N} X_i \Rightarrow d_{TV}(S_N, \mathrm{Poi}(\lambda)) \leq 2\lambda^2/N$

(i) $\Rightarrow$ (iv) For any $t \geq 0$, by definition (i) we have that $N(t) \sim \mathrm{Poi}(\lambda t)$. To show (iv), we need to show the following.

Thm - Given $N(t) = n$, the n arrival times $T_1, T_2, \ldots, T_n$ have the same distribution as the order statistics of n independent rv $X_i \sim U[0,t]$.

Pf - Let $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ denote $X_1, X_2, \ldots, X_n$ in sorted order, and consider the event $\mathcal{E}_s = \{Y_{(1)} \leq s_1, Y_{(2)} \leq s_2, \ldots, Y_{(n)} \leq s_n\}$ for any $0 < s_1 < s_2 < \cdots < s_n < t$. Similarly, for any permutation $\sigma = (\sigma(1), \sigma(2), \ldots, \sigma(n))$ of $[n]$, let

$\mathcal{E}_{\sigma,s} = \{X_{\sigma(1)} \leq s_1, X_{\sigma(1)} < X_{\sigma(2)} \leq s_2, \ldots, X_{\sigma(n-1)} \leq X_{\sigma(n)} \leq s_n\}$

Then the events $\mathcal{E}_\sigma$ are disjoint for any $s = \{s_i\}$, and $\mathcal{E}_s = \bigcup_{\sigma:\text{ permutation of }[n]} \mathcal{E}_{\sigma,s}$. Thus we have

$$
\begin{aligned}
\mathbb{P}[\mathcal{E}_s] &= \mathbb{P}\left[\bigcup \mathcal{E}_{\sigma,s}\right] = \sum_\sigma \mathbb{P}[\mathcal{E}_{\sigma,s}] \\
&= \sum_\sigma \mathbb{P}\left[X_{\sigma(1)} \leq s_1, X_{\sigma(2)} < X_{\sigma(1)} \leq s_2, \ldots, X_{\sigma(n-1)} < X_{\sigma(n)} \leq s_n\right] \\
&= n!\, \mathbb{P}\left[X_1 \leq s_1, X_1 < X_2 \leq s_2, \ldots, X_{n-1} < X_n \leq s_n\right] \\
&= n! \int_0^{s_1} \int_{x_1}^{s_2} \int_{x_2}^{s_3} \cdots \int_{x_{n-1}}^{s_n} \left(\frac{1}{t}\right)^n dx_n\, dx_{n-1} \cdots dx_1 \\
&= \int_{[0,t]^n} f(x_1, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n, \quad \text{where } f(x_1, \ldots, x_n) = \frac{n!}{t^n}
\end{aligned}
$$

This is the joint pdf of the order statistics of n $U[0,1]$ rv $\boxtimes$

Properties of the Poisson Process

- There are 3 other important properties that characterize Poisson processes.
- First, note that if $X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$, $X \perp\!\!\!\perp Y$, then $Z = X + Y \sim Poi(\lambda+\mu)$. To see this, note that

$$\mathbb{P}[Z=k] = \sum_{i=0}^{k} \mathbb{P}[X=i, Y=k-i] = \sum_{i=0}^{k} \frac{e^{-\lambda}\lambda^i e^{-\mu}\mu^{k-i}}{i!\,(k-i)!} = \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^k}{k!}$$

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1) Superposition of PP - Let $\{N_i\}_{i\geq 1}$ be a collection of independent PP with rates $\{\lambda_i\}_{i\geq 1}$. Then
   - $\mathbb{P}[N_i, N_j$ both have a jump at $t] = 0$ a.s. $\forall i, j, t$
   - If $\lambda \triangleq \sum_{i=1}^{\infty} \lambda_i < \infty$, then $N(t) \triangleq \sum_{i=1}^{\infty} N_i(t)$ is a $PP(\lambda)$

Pf - $\forall t \geq 0$, $\mathbb{E}[N(t)] = \mathbb{E}[\sum_{i=1}^{\infty} N_i(t)] = \lambda t < \infty$, $N(0)=0$

$\Rightarrow N(t_2) - N(t_1)$ is a.s. finite for all $(t_1, t_2] \subset \mathbb{R}$.

Moreover $N(t_2)-N(t_1) \perp\!\!\!\perp N(t_4)-N(t_3)$ $\forall$ disjoint intervals since each is a sum of independent r.v.s

Finally

$$\mathbb{P}[N(t_2)-N(t_1)=k] = \lim_{n\uparrow\infty} \mathbb{P}\left[\sum_{i=1}^{n}(N_i(t_2)-N_i(t_1))=k\right]$$

$$= \lim_{n\uparrow\infty} \frac{e^{-(\sum_{i=1}^{n}\lambda_i(t_2-t_1))}\left(\sum \lambda_i(t_2-t_1)\right)^k}{k!}$$

$$= e^{-\lambda(t_2-t_1)}\left(\lambda(t_2-t_1)\right)^k / k!$$

2) Random Splitting of PP - If $N(t)$ is a $PP(\lambda)$, and it is split randomly into $N(t) = A_1(t) + A_2(t)$, wherein each arrival to $N(t)$ is independently sent to $A_1(t)$ w.p. p, else to $A_2(t)$. Then $A_1(t) \sim PP(p\lambda)$, $A_2(t) \sim PP((1-p)\lambda)$ and $A_1(t) \perp\!\!\!\perp A_2(t)$.

Pf - Clearly $A_1(0) + A_2(0) = 0$. Moreover, we know that for $N(t)$, for any t, δ we have $\mathbb{P}[N(t+\delta) - N(t) = 0] = \lambda\delta + o(\delta^2)$, $\mathbb{P}[N(t+\delta) - N(t) = 1] = 1 - \lambda\delta + o(\delta^2)$ and $\mathbb{P}[N(t+\delta) - N(t) > 1] = o(\delta^2)$.

By construction, we have $\mathbb{P}[A_1(t+\delta) - A_1(t) = 1] = \lambda\delta p + o(\delta^2)$, $\mathbb{P}[A_1(t+\delta) - A_1(t) > 1] = o(\delta^2)$ and $\mathbb{P}[A_1(t+\delta) - A_1(t) = 0] = 1 - \lambda\delta p + o(\delta^2)$ (and similarly for $A_2(t)$). Thus we see $A_1(t) \sim PP(\lambda p)$ and $A_2(t) \sim PP((1-p)\lambda)$. Finally to see $A_1(t) \perp\!\!\!\perp A_2(t)$, note that

$$\mathbb{P}[(A_1(t) = m) \cap (A_2(t) = n)] = \mathbb{P}[(N(t) = n+m) \cap (A_1(t) = m)]$$

$$= \frac{e^{-\lambda t} (\lambda t)^{m+n}}{(m+n)!} \cdot \binom{m+n}{m} p^m (1-p)^n$$

$$= \frac{e^{p\lambda t} e^{(1-p)\lambda t} (\lambda t p)^m (\lambda t (1-p))^n}{m!\, n!}$$

$$= \mathbb{P}[A_1(t) = m]\, \mathbb{P}[A_2(t) = n]$$

Note - Let $X_i \sim \mathcal{E}xp(\lambda_i)$ $\perp\!\!\!\perp$, $\sum_{i=1}^{n} \lambda_i = \lambda < \infty$, and $U = \min\{X_i\}$.
Then $U \sim \mathcal{E}xp(\lambda)$. To see this, note

$$\mathbb{P}[U \geq a] = \mathbb{P}\left[\bigcap_{i=1}^{n} X_i \geq a\right] = \prod_{i=1}^{n} \mathbb{P}[X_i \geq a]$$
$$= \prod_{i=1}^{n} e^{-\lambda_i a} = e^{-\lambda a}$$

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3) Competition among PP - If $N(t) = \sum_{i=1}^{\infty} N_i(t)$, where $N_i \sim PP(\lambda_i)$, independent, and $T_{1,i}$ is the first arrival time for $N_i(t)$ and $T_1 = \min_i \{T_{1,i}\}$ the first arrival for $N(t)$. Also let $J = \operatorname{argmin}_i \{T_{1,i}\} \equiv$ first arriving process.
Then

$$\mathbb{P}[T_1 \geq a, J = i] = \mathbb{P}[T_1 \geq a]\,\mathbb{P}[J = i] \qquad (\lambda = \textstyle\sum \lambda_i < \infty)$$
$$= \left(e^{-\lambda a}\right)\left(\lambda_i / \lambda\right)$$

Pf - First consider $N(t) = \sum_{i=1}^{K} N_i(t)$ for finite K

$$\mathbb{P}[T_1^{(K)} \geq a] = \mathbb{P}\left[\bigcap_{i=1}^{K} \{N_i(t) \geq a\}\right] = \prod_{i=1}^{K} e^{-\lambda_i a} = e^{-\lambda a}$$

Moreover

$$\mathbb{P}[J^{(K)} = i, T_1^{(K)} \geq a] = \mathbb{P}\left[a \leq T_{1,i} \leq \inf_{j \neq i}(T_{1,j})\right]$$
$$= \int_a^{\infty} \lambda_i e^{-\lambda_i x}\, \mathbb{P}\left[\inf_{j \neq i}(T_{1,j}) > x\right] dx$$
$$= \int_a^{\infty} \lambda_i e^{-\lambda_i x}\, e^{-\sum_{j \neq i} \lambda_j}\, dx$$
$$= \left[\lambda_i / \left(\textstyle\sum_j \lambda_j\right)\right] e^{-(\sum_j \lambda_j) a}$$

Finally since $\{T_1^{(K)} \geq a, J^{(K)} = i\} \searrow \{T_1 \geq a, J = i\}$, we can take limits $\blacksquare$

Finally, we can generalize the basic PP in 2 ways.

- Time-inhomogeneous Poisson process - Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a right-continuous fn with $\int_0^t \lambda(x)dx < \infty \ \forall t$.

Then $N(t)$ is a non-homogeneous PP with rate $\lambda(t)$ if $N(0)=0$, and for any $k \in \mathbb{N}$, and any $0 < t_1 < t_2 < \dots < t_k$,

$N(t_i) - N(t_{i-1}) \perp\!\!\!\perp N(t_j) - N(t_{j-1}) \ \forall i, j \in [k], i \neq j$, and

$$N(t_i) - N(t_{i-1}) \sim \mathrm{Poi}\left(\int_{t_{i-1}}^{t_i} \lambda(x)dx \right)$$

- Spatial Poisson point process - Let $N = \{N((x,y)), (x,y) \in \mathbb{R}^2\}$ be a collection of random points in $\mathbb{R}^2$ s.t:

i) $\forall B \subset \mathbb{R}^2$ with Lebesgue measure $|B|$, the number of points $N(B) = \int_B N(x,y)\,dx\,dy$ satisfies

$$\mathbb{P}[N(B) = n] = (\lambda|B|)^n e^{-\lambda|B|}/n!, \quad \forall n \in \mathbb{N}$$

ii) $\forall B_1, B_2, \dots, B_k \subset \mathbb{R}^2$, $B_i \cap B_j = \phi \ \forall i,j \in [k]$, we have $N(B_i) \perp\!\!\!\perp N(B_j)$