A <ins>coupling of a Markov Chain</ins> (Ω, P) is a pair of processes $(X_t, Y_t)_{t=0}^{\infty}$ s.t.

i) both X_t, Y_t are MC on Ω with transition matrix P

ii) $X_t = Y_t \Rightarrow X_{t+1} = Y_{t+1}$ (coalescence)

- If $X_0 = x$, $Y_0 = y$, then we denote $\mathbb{P}_{x,y}$, $\mathbb{E}_{x,y}$ to be w.r.t the probability space on which X_t, Y_t are jointly defined.

- $\boxed{\text{Thm}}$. Given Markov Chain P and coupling (X_t, Y_t) with $X_0 = x$, $Y_0 = y$. Let $\tau_{couple} = \min\{t \mid X_t = Y_t\}$

Then
$$\underbrace{\| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV}}_{d_{xy}(t)} \leq \mathbb{P}_{x,y}[\tau_{couple} > t]$$

Pf. $P^t(x,z) = \mathbb{P}_{x,y}[X_t = z]$, $P^t(y,z) = \mathbb{P}_{x,y}[Y_t = z]$ $\forall z \in \Omega$

- This means that X_t, Y_t form a coupling for $P^t(x,\cdot), P^t(y,\cdot)$
- Thus $\mathbb{P}_{x,y}[X_t \neq Y_t] \geq \| P^t(x,\cdot) - P^t(y,\cdot) \|_{TV}$
- Finally, note $\mathbb{P}_{x,y}[X_t \neq Y_t] = \mathbb{P}_{x,y}[\tau_{couple} > t]$

- Since for any x, y, we have $d_{xy}(t) \leq \mathbb{P}_{x,y}[\tau_{couple} > t]$

$$\Rightarrow d(t) \leq \boxed{\bar{d}(t) = \max_{x,y \in \Omega} \mathbb{P}_{x,y}[\tau_{couple} > t]}$$

Eg- Lazy RW on $\{0,1\}^n$

- $P \equiv$ wp 0.5, $X_{t+1} = X_t$
  wp 0.5, Pick ~~random~~ u.a.r coordinate, flip bit
- (Check) P is irreducible, aperiodic
- (Check) π = Uniform over $\{0,1\}^n$
- To bound $t_{mix}(1/2e)$: Consider foll^g coupling
  i) Pick coordinate c u.a.r
  ii) If $X_t[c] = Y_t[c]$, then stay $X_{t+1}[c] = Y_{t+1}[c] = X_t[c]$ wp 0.5
  flip $X_{t+1}[c] = Y_{t+1}[c] = \overline{X_t[c]}$ wp 0.5
  iii) If $X_t[c] \neq Y_t[c]$, then set $X_{t+1}[c] = Y_{t+1}[c] = 0$ wp 0.5 / 1 wp 0.5
- (Check) This is a valid coupling (i.e, $X_t, Y_t \sim P$)
- $\tau_{couple} \equiv$ Coupon collector problem with n bins

$$\Rightarrow \mathbb{P}[\tau_{couple} > n \ln(n) + cn] \leq e^{-c}$$

$$\Rightarrow t_{mix}(1/2e) \leq n(\ln(n) + \ln(2e)) = O(n\ln(n))$$

Now, instead of a self-loop prob of ½, consider the following walk on $\{0,1\}^n$ ≡ i) pick $i \in \{0,1,\dots,n\}$
ii) If $i=0$, do nothing
else, flip bit i of X_t

$\Rightarrow$ self loop $= 1/_{n+1}$

- Coupling for this chain ≡ Let $d_H(X_t, Y_t) \equiv$ Hamming distance betn X_t, Y_t

if $d(X_t, Y_t) > 1$.

i) ~~If X_t picks~~ Pick $i \in \{0,1,\dots,n\}$ uar

ii) If $i=1$, do nothing to both X_t, Y_t

iii) If $X_t[i] = Y_t[i]$, do nothing

iv) If $X_t[i] \neq Y_t[i]$, then flip $X_t[i]$ and $Y_t[f(i)]$, where $f(i)$ is an (arbitrary) cyclic permutation on disagreeing bits of X_t, Y_t

v) If $d(X_t, Y_t) = 1$ and X_t and Y_t disagree on i_0
- If X_t picks i_0, Y_t picks 0 (and vice versa)

- $d_{Hamming}(X_t, Y_t)$ is non-increasing.

- Equiv to coupon collector on $n/2$ coupons!

$$t_{mix}(1/2e) \leq \frac{1}{2} n \ln(n) + O(n)$$

- Mixing time for random transposition shuffle

- Pick 2 cards at random (with replacement) and swap
- Equivalently - pick card c, position $p \in \{1,2,\dots,n\}$ uar
  - exchange card c with card in position p
- Coupling $(X_t, Y_t) \equiv$ Pick same c, p in all steps
- Let $d(X_t, Y_t) =$ # of positions in which X_t, Y_t differ

Dynamics of $d_t = d(X_t, Y_t) \equiv$ i) If c at same pos^n in $X_t, Y_t \Rightarrow d_{t+1} = d_t$
ii) If c at diff pos^{n}s in X_t, Y_t
- If card at pos^n P same $\Rightarrow d_{t+1} = d_t$
- If card at pos^n P diff $\Rightarrow d_{t+1} \leq d_t - 1$

$$\Rightarrow \mathbb{P}\left[d_{t+1} \leq d_t - 1\right] = \left(\frac{d_t}{n}\right)\cdot\left(\frac{d_t}{n}\right) \Rightarrow \mathbb{E}\left[\text{Time for } d_t \to d_t - 1\right] \leq \left(\frac{n}{d_t}\right)^2$$

$$\Rightarrow \mathbb{E}\left[\tau_{couple}\right] \leq \sum_{d_t = n}^{1} \left(\frac{n}{d_t}\right)^2 \leq \frac{6n^2}{\pi^2}$$

$$\Rightarrow \mathbb{P}\left[\tau_{couple} > t\right] \leq \frac{6n^2}{\pi^2 t} \Rightarrow t_{mix}(1/2e) \leq \frac{3n^2}{\pi^2 e} = O(n^2)$$

- The true mixing time is $O(n \log n)$

- **Grand Coupling** - Common source of randomness used to construct coupling for every starting state $x \in \Omega$
  - the previous example was a grand coupling
  - special case of **Markovian coupling** - Joint transition matrix Q defines a MC over $\Omega \times \Omega$ s.t
    i) $\forall x, y, x'$, $\sum_{y'} Q((x,y),(x',y')) = P(x,x')$, ii) Same for x, y, y'

* Metropolis-Hastings for sampling graph colorings

- $\boxed{\text{Graph coloring}}$ - Given $G(V,E)$ graph, $Q=\{1,\dots,q\} \equiv$ set of colors
  - Δ = max degree of G
  - Goal: Sample uar $c \in Q^{|V|}$ s.t
    $c(u) \neq c(v) \;\forall (u,v) \in E$

- Do proper colorings always exist for given G?
  - If $q \geq \Delta+1$, YES (prove this)
  - If $q = \Delta \equiv$ (Brook's Thm) - If $\Delta=2$, then YES iff no $(\Delta+1)$ clique and no odd cycle $K_{\Delta+1}$
    - If $\Delta>2$, YES iff no $K_{\Delta+1}$
  - If $q < \Delta \equiv$ NP hard to decide

- MH for sampling colorings $\equiv$
  i) Start at arbitrary legal coloring
  ii) Pick vertex V_t, color C_t uar
  iii) If $c(V_t) = C_t$ is valid, replace
  Else do nothing

- This is clearly reversible (symmetric), aperiodic. Not irreducible if $q=\Delta+1$ (consider $K_{\Delta+1}$).

- **Claim** - MH chain irreducible if $q > \Delta + 1$ (check)

- $\boxed{\text{Thm}}$ - If $q \geq 4\Delta + 1$, then mixing time is $O(n\log n)$

Pf - Coupling (X_t, Y_t) defined as follows-

i) Pick (V_t, C_t) uar and apply to (X_t, Y_t)

- Let $d_t = d(X_t, Y_t) = \#$ of vertices where colors disagree in X_t and Y_t

- Good Moves $\equiv d_t \to d_t - 1$
  - Happens if ~~...~~ V_t has different colors in X_t, Y_t and C_t is a valid color (i.e, not in nbd of V_t)
    - $\mathbb{P}[\text{Good Move}] \geq \dfrac{(q-2\Delta) \cdot d_t}{nq}$

- Bad Moves $\equiv d_t \to d_t + 1$
  - Happens if V_t is agreeing, and C_t is such that it is valid for X_t, invalid for Y_t, or vice versa
    - $\mathbb{P}[\text{Bad Move}] \leq \dfrac{1}{nq} \cdot \underbrace{2}_{\text{Choose } X_t \text{ or } Y_t} \cdot \underbrace{\Delta d_t}_{\# \text{ of vertices in nbd of disagreeing vertex}}$

bound on valid C_t

- Neutral Moves $\equiv d_t \to d_t$

- $$\mathbb{E}\left[d_{t+1} \mid X_t, Y_t\right] \leq d_t - \frac{d_t(q-2\Delta)}{qn} + \frac{d_t \cdot 2\Delta}{qn}$$

$$= d_t\left(1 - \frac{q-4\Delta}{qn}\right)$$

$$\Rightarrow \mathbb{E}\left[d_t \mid X_0, Y_0\right] \leq d_0\left(1 - \frac{q-4\Delta}{qn}\right)^t$$

$$\leq n\left(1 - \frac{q-4\Delta}{qn}\right)^t \leq n\exp\left(\frac{4\Delta - q}{qn}\right)$$

$$\Rightarrow t_{mix}(\varepsilon) \geq \left(\frac{q}{q-4\Delta}\right) n \left(\log n + \log 1/\varepsilon\right)$$

$$\left(\text{Since } \mathbb{P}\left[X_t \neq Y_t \mid X_0, Y_0\right] = \mathbb{P}\left[d_t \geq 1 \mid X_0, Y_0\right] \leq \mathbb{E}\left[d_t \mid X_0, Y_0\right]\right)$$

$$\Rightarrow \boxed{t_{mix}\left(1/2e\right) \leq \left(\left(\frac{q}{q-4\Delta}\right) n\left(\ln(n) + \ln(2e)\right)\right)}$$