Perfect Sampling

- We will see two methods of generating perfect samples $X \sim \pi$ for given markov chain P

i) Strong stationary times ii) Coupling from the past

Eg - (Top-to-random shuffle) - Given n cards

- — take top card and insert it u.a.r in position $\{1, 2, .., n\}$
  — Random walk on $S_n \equiv$ Symmetric group (i.e., set of permutations of $[n]$)
  — $\sigma_t \equiv$ Permutation at time t
- Let $\tau_{top} = 1 + \min\{t \geq 0 \mid \sigma_t(1) = \sigma_0(n)\}$
  (i.e., first time bottom card comes to top plus 1)
- **Propn** - If at time t, k cards below original bottom card, then these are perfectly shuffled. Hence $\sigma_{\tau_{top}} \equiv$ Uniform in S_n

Pf - True at $t=0$. Suppose its true at time t; then in σ_{t+1}, either # of cards below original bottom card is same, or new card inserted at random position below original bottom card.

- Mixing time $\equiv$ Coupon collector

- Stopping time T is a -

i) Stationary time if $\mathbb{P}_x[X_T = y] = \pi(y) \ \forall x, y$

ii) <u>Strong stationary time</u> if $\underbrace{\mathbb{P}_x[T=t, X_T=y] = \mathbb{P}_x[T=t]\pi(y)}_{\text{or } \mathbb{P}_x[X_t=y \mid T=t] = \pi(y)}$

(i.e., X_T has distn π <u>and</u> $T \perp\!\!\!\perp X_T$)

- <u>Lemma</u> - Let $X_t \sim MC(\Omega, P)$ with stationary distn π. If τ_{st} is a strong stopping time for P, then

$$d(t) \left(= \max_{x\in\Omega} \|P^t(x,\cdot) - \pi\|_{TV}\right) \leq \max_{x\in\Omega} \mathbb{P}_x[\tau_{st} > t]$$

<u>Pf</u> - For any $x \in \Omega$, $d_x(t) = \max_{A\subseteq\Omega} \|P^t(x,A) - \pi(A)\|_{TV}$. Now

$$\mathbb{P}_x[X_t \in A] = \mathbb{P}_x[X_t\in A, \tau_{st} > t] + \sum_{t'\leq t} \mathbb{P}_x[X_t\in A, \tau_{st}=t]$$

$$= \mathbb{P}_x[X_t\in A \mid \tau_{st}>t]\,\mathbb{P}_x[\tau_{st}>t] + \pi(A)\sum_{t'\leq t}\mathbb{P}[\tau_{st}=t]$$

$$= \mathbb{P}_x[X_t\in A \mid \tau_{st}>t]\,\mathbb{P}_x[\tau_{st}>t] + \pi(A)(1 - \mathbb{P}_x[\tau_{st}>t])$$

$$\Rightarrow \mathbb{P}_x[X_t\in A] - \pi(A) = \mathbb{P}_x[\tau_{st}>t]\left(\underbrace{\mathbb{P}[X_t\in A \mid \tau_{st}>t] - \pi(A)}_{\in[-1,1]}\right)$$

This holds $\forall x\in\Omega$, $A\subseteq\Omega$

$$\Rightarrow \quad d(t) \leq \max_{x\in\Omega}\mathbb{P}_x[\tau_{st}>t]$$

<u>Note</u> - Independence of τ_{st} and $X_{\tau_{st}}$ is critical!

<u>Eg</u> - For RW on n-cycle, let $\overset{Z}{\tau=0 \text{ and }} = Ber(1-1/n)$. If $Z=0$, then$_n$ $X_\tau \triangleq 0$; else, run RW from 0 and let τ = cover time.

<u>Eg</u> - For RW on hypercube $\{0,1\}^n$, consider following chain
- Pick I_t uar from $\{1,2,\dots,n\}$
- Flip $X_t(I_t)$ wp ½

<u>Claim</u> - τ_{st} = First time $\{I_1, I_2, \dots, I_t\} = \{1,2,\dots,n\}$
- Thus $\tau_{st} \equiv$ Coupon collector time
- $\mathbb{P}_x[\tau_{st} > n\ln n + cn] \leq e^{-c}$
  $\Rightarrow t_{mix}(\varepsilon) \leq n(\ln(n) + \ln(1/\varepsilon))$
- Identical to coupling time of random walks.

<u>Eg</u> - Transposition Shuffle (Pick L_t, R_t uar with replacement & swap)
- Earlier we showed $t_{mix}(\varepsilon) \leq \frac{6n^2}{\pi^2 \varepsilon}$ via mixing arguments
- SS time (Broder) - Start with no marked card
  - (In round t, mark R_t if unmarked) <u>and</u> (L_t marked <u>OR</u> $L_t = R_t$)

(Sec 8.2 in LPW)

- $\mathbb{E}[\tau_{st}] = \sum_{k=0}^{n-1} \frac{n^2}{(k+1)(n-k)} = 2n(\log n + O(1))$, $Var(\tau_{st}) = O(n^2)$

By Chebyshev - $t_{mix}(\varepsilon) \leq n \log n \cdot (2 + 9\sqrt{\varepsilon})$

Random Mapping Representation of ~~a~~ MC

- Given MC (Ω, P), and an independent Λ-valued random variable Z ~~satisfying~~, a random mapping representation is a fn $f : \Omega \times \Lambda \to \Omega$ s.t.

$$\mathbb{P}[f(x|\Lambda) = y] = P(x,y)$$

- Eg - i) RW on n-cycle - $Z \sim \pm 1$ wp $1/2$

  - $f(x|Z) = (x+Z) \bmod n$

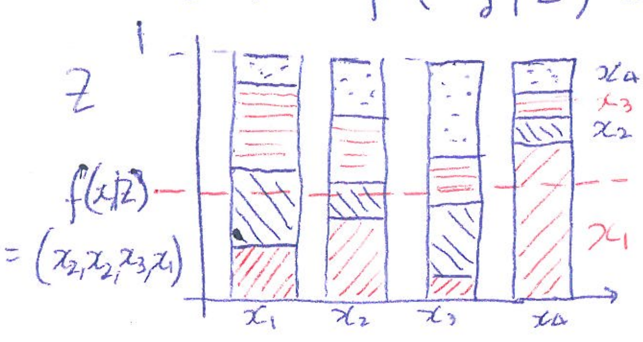
  ii) Lazy RW on hypercube - $Z \sim \{\underbrace{Unif(\{1,2,\dots,n\})}_{I}, \underbrace{Ber(1/2)}_{F}\}$

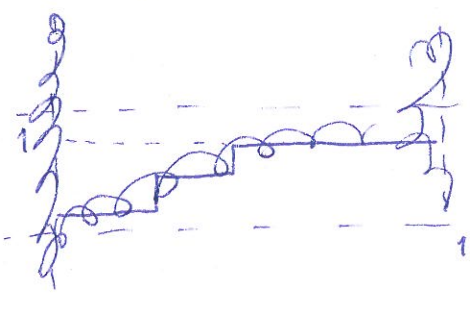
  - $f(x|Z) \equiv$ flip $x(I)$ if $F=1$, else no change

- Lemma: Every P on finite Ω has a random mapping representation

Pf - - Choose arbitrary ordering $\Omega = \{x_1, x_2, \dots, x_n\}$

- Generate $Z \sim Unif([0,1])$
- Define $F_{j,k} = \sum_{i=1}^{k} P(x_j, x_i)$
- Set $f(x_j|Z) = x_k$ when $F_{j,k-1} < Z \le F_{j,k}$





- Some obs on random mapping representations
  - Not unique
  - $\{f(x|Z)\}_x$ forms a grand coupling (i.e, $(f(X_t|Z), f(Y_t|Z))$ is a coupling for P
  - If $|\Omega| = n$, then sufficient to consider a discrete r.v Z with $\leq n^2$ values

    (Corresponding to breakpoints $F_{j,k}$ $\forall (j,k) \in \Omega^2$)
  - If $(X_t, Y_t) \sim (f(X_{t-1}|Z_t), f(Y_{t-1}|Z_t))$, then

    τ_{couple} related to $\max_{x,y} \| f(x|Z) - f(y|Z) \|_{TV}$

    (or more generally $\| f(f(\cdots f(x|Z_1)|Z_2)\cdots)|Z_k) - f(f(\cdots f(y|Z_1)|Z_2)\cdots|Z_k) \|_{TV}$
    - Henceforth write $\underbrace{f \circ f}(x) = f(f(x|Z_1)|Z_2)$

      composition of random fns
  - Can obtain a SS time from a random function representation (with $Z \in \Lambda$, $|\Lambda| \leq n^2$) by

    No ⟶ sampling $Z_1, Z_2, \ldots Z_{\tau_{st}}$ until we hit every value in Λ (i.e, coupon collector on Λ)
    - Problem - Difficult to specify Λ in general

Eg - MC (Ω, P) s.t $\alpha = \sum_{y} \min_{x} [P(x,y)] > 0$

(i.e, sum over states of min transition prob)

- Related to the strong Doeblin condition

- Can write $P = \alpha\, 1^T \theta + (1-\alpha) Q$,

where $\theta \equiv$ distn over Ω and $Q \equiv$ stochastic matrix

- Natural induced coupling - wp α, ~~sample~~ set $X_t = Y_t \sim \theta$
  - else, advance (X_t, Y_t) using Q

This is a random mapping representation!

- $\mathbb{P}[\tau_{couple} > t] \leq (1-\alpha)^t$

$\Rightarrow t_{mix}(\varepsilon) \leq \frac{1}{\varepsilon} \ln(1-\alpha)$

- To generate a perfect sample from π
  - Sample $X_0 \sim \theta$, $\tau \sim Geom(\alpha)$
  - Return $X_{(\tau-1)}$

Pf - $\pi P = \alpha \underbrace{\pi 1^T \theta}_{\theta} + (1-\alpha)\pi Q = \pi$

$\Rightarrow \pi = (I - (1-\alpha) Q)^{-1} \alpha \theta = \sum_{k=0}^{\infty} (1-\alpha)^k \alpha Q^k \theta$

$= \mathbb{E}[Q^{\tau-1} \theta]$