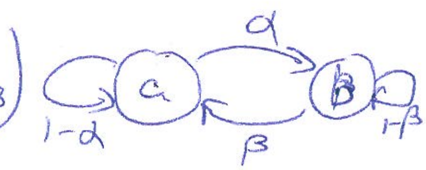
Spectral techniques & Multicommodity Flows

- First lecture - we explicitly worked out a simple 2-state example - $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$, $\pi = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$



$$P^n = \begin{pmatrix} \beta/\alpha+\beta & \alpha/\alpha+\beta \\ \beta/\alpha+\beta & \alpha/\alpha+\beta \end{pmatrix} + \underbrace{(1-\alpha-\beta)}_{\text{e-value of } P}{}^n \begin{pmatrix} \alpha/\alpha+\beta & -\alpha/\alpha+\beta \\ -\beta/\alpha+\beta & -\beta/\alpha+\beta \end{pmatrix}$$

Now we generalize this to general P

- $\boxed{\text{Thm (Perron-Frobenius)}}$

For any $\underbrace{\text{non-negative}}_{\text{i.e., } A_{ij} \geq 0 \ \forall i,j}$, $\overbrace{\text{primitive}}^{\text{i.e., } A^k > 0 \text{ for some } k \text{ integer}}$ (equiv to irreducible + aperiodic) $n \times n$ matrix A, $\exists$ a real eigenvalue λ_1 (with algebraic and geometric multiplicity 1) such that

i) $\lambda_1 > 0$, $\lambda_1 > |\lambda_j|$ for any other e-value λ_j

ii) can choose left e-vector u_1, right e-vector v_1 s.t $u^T v = 1$

iii) If A is stochastic, then $\lambda_1 = 1$, $v_1 = \mathbb{1}$, $u_1 = \pi$

iv) If A is not irreducible, then multiplicity of e-value 1 is equal to # of communicating classes

v) If A has period d, then there are d e-values with modulus $= 1$

Corollary - For any aperiodic, irreducible MC (Ω, P)

$$P^k = \mathbb{1}^T \pi + O(k^{m_2-1} \underbrace{|\lambda_2|^k}_{\text{SLEM}})$$

SLEM - Second largest e-value modulus

- $\lambda_j, j>2$ may be complex for general MC...

- <u>E-values of Reversible ~~Matrices~~ MCs</u>

- Reversible $\Leftrightarrow$ $\pi(i) P(i,j) = \pi(j) P(j,i) \quad \forall i,j \in \Omega$

- Given (Ω, P, π), define vector space $\ell^2(\pi)$ as the space $\mathbb{R}^n$ endowed with inner product

$$\langle f, g \rangle_\pi = \sum_{x \in \Omega} f(x) g(x) \pi(x)$$

- Lemma - (P, π) is reversible iff <u>P is self-adjoint</u> in $\ell^2(\pi)$.

i.e., $\langle Pf, g \rangle_\pi = \langle f, Pg \rangle_\pi \quad \forall f, g$.

<u>Pf</u> - $\langle Pf, g \rangle_\pi = \sum_{x \in \Omega} \left(\sum_{y \in \Omega} P(x,y) f(y) \right) \pi(x) g(x)$

$$= \sum_{x,y \in \Omega^2} P(x,y) \pi(x) g(x) f(y) = \sum_{x,y \in \Omega^2} P(y,x) \pi(y) g(x) f(y)$$

$$= \sum_{y \in \Omega} f(y) \pi(y) \left(\sum_{x \in \Omega} g(x) P(y,x) \right) = \langle f, Pg \rangle_\pi$$

- For only if, choose $f(\cdot) = e_x$, $g(\cdot) = e_y$ to get $\pi(x) P(x,y) = \pi(y) P(y,x)$

- Consider $P^* = D^{1/2} P D^{-1/2}$, $D = \mathrm{diag}(\pi)$
  - P reversible $\Leftrightarrow$ P^* symmetric
  - Also P, P^* have same e-values $\Rightarrow$ $\lambda_1, \lambda_2, \ldots, \lambda_n$ are <u>real</u>

  (Recall by PF thm, P irreducible $\Rightarrow \lambda_1 = 1$, $\max\{|\lambda_2|, |\lambda_n|\} < 1$)

  - Let $w_1, w_2, \ldots, w_n \equiv$ <u>orthonormal basis</u> for P^*, $u = D^{-1/2} w$, $v = D^{1/2} w$
    (spectral thm for symmetric matrices)

    $\Rightarrow$ $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are left and right e-vectors of P and $\langle u_i, u_j \rangle_{\frac{1}{\pi}} = \langle v_i, v_j \rangle_\pi = \delta_{ij}$ (<u>orthonormal in $\ell^2(\pi)$</u>)

  - Thus, any $x \in \mathbb{R}^n \equiv x = \sum_{j=1}^{n} \langle x, v_j \rangle_\pi v_j$

    $$\Rightarrow \boxed{P^k f = \sum_{j=1}^{n} \lambda_j^k \langle f, v_j \rangle_\pi v_j} = \langle f, \mathbb{1} \rangle_\pi \mathbb{1} + \sum_{j=2}^{n} \lambda_j^k \langle \cdot, \cdot \rangle v_j$$

    Choose $f(\cdot) = \delta_y$ to get $P^k(x,y) = \pi(y) + \sum_{j=2}^{n} \lambda_j^k \pi(y) v_j(x) v_j(y)$

- $$\gamma^* = 1 - \overbrace{\max\{|\lambda_2|, |\lambda_n|\}}^{\lambda_*} \qquad (\text{where } 1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n)$$

  (SLEM or absolute spectral gap)

<u>Thm</u>. (Ω, P, π) irreducible MC, $\pi_{\min} = \min_{x \in \Omega} \pi(x)$, $\gamma^* = \text{SLEM}$

then $$t_{mix}(\varepsilon) \leq \frac{1}{\gamma^*} \ln\left(\frac{1}{\varepsilon \pi_{\min}}\right)$$

and $$t_{mix}(\varepsilon) \geq \left(\frac{1}{\gamma^*} - 1\right) \ln\left(\frac{1}{2\varepsilon}\right)$$

Pf - $\left|\frac{P^k(x,y)}{\pi(y)} - 1\right| = \left|\sum_{j=2}^{n} \lambda_2^k v_j(x) v_j(y)\right|$

$$\leq \lambda_*^k \left[\sum_{j=2}^{n} v_j^2(x) \sum_{j=2}^{n} v_j^2(y)\right]^{1/2} \quad \text{(Cauchy Schwarz)}$$

Recall $v_j = D^{1/2} w_j$, w_j orthonormal $\Rightarrow \sum_{j=1}^{n} v_j^2(x) \leq 1/\pi(x)$

(Alt, $\pi(x) = \langle \delta_x, \delta_x \rangle_\pi = \left\langle \sum_{j=1}^{n} v_j(x)\pi(x)\, v_j, \sum_{j=1}^{n} v_j(x)\pi(x)\, v_j \right\rangle_\pi$

$= \sum_{j=1}^{n} v_j^2(x) (\pi(x))^2$, via $\langle v_i, v_j\rangle_\pi = \delta_{ij}$)

Also, $\|P^k(x,\cdot) - \pi\|_{TV} \leq \max_y \left[1 - \frac{P^k(x,y)}{\pi(y)}\right]$ (check)

$\Rightarrow$ $d(k) \leq \frac{\lambda_*^k}{\pi_{min}} \leq \frac{e^{-\gamma^* k}}{\pi_{min}}$

- For lower bound, consider v_i s.t $Pv_i = \lambda_i v_i$, $i > 1$

Also $\langle 1, v_i \rangle_\pi = 0$ (orthogonality of v_i in $\ell^2(\pi)$)

$\Rightarrow |\lambda_i^k v_i(x)| = |P^k v_i(x)| = \left|\sum_{y \in \Omega} [P^k(x,y) v_i(y) - \pi(y) v_i(y)]\right|$

$\leq \|v_i\|_\infty \cdot 2d(k)$

Now we can choose x s.t $\|v_i\|_\infty = v_i(x) \Rightarrow |\lambda_i^k| \leq 2d(k)$

$\Rightarrow |\lambda^*|^{t_{mix}(\varepsilon)} \leq 2\varepsilon \Rightarrow t_{mix}(\varepsilon) \ln(1/\lambda^*) \geq 2\varepsilon$

- **Corollary** - (P, π) reversible, irreducible $\Rightarrow \lim_{t\to\infty} (d(t))^{1/t} = \lambda_*$

- **Note** - For lazy chain, $\lambda_i \geq 0\ \forall i \Rightarrow \lambda^* = \lambda_2$
  (ie, $\hat{P} = \frac{1}{2}(I+P)$ = psd!)

- **Eg** - For RW on n-cycle, $\lambda_{j+1} = \cos(2\pi j/n)$. Assume n=odd

$\Rightarrow \lambda^* = \cos(2\pi/n) = 1 - \frac{4\pi^2}{2n^2} + O(n^{-4})$, $\gamma^* = \Theta(n^{-2})$

<u>Note</u> - if n = even, then -1 is an e-value. Aperiodic

- For lazy RW on hypercube, $\lambda_{j+1} = \frac{n-j}{n}$, $\lambda^* = 1 - \frac{1}{n}$, $\gamma^* = \frac{1}{n}$

(with multiplicities)

- For any $f : \Omega \to \mathbb{R}$
  - $\mathrm{Var}_\pi(f) = \sum_x \pi(x)\left(f(x) - \sum_y \pi(y) f(y)\right)^2 = \sum_x \left(\pi(x) f(x) \sum_y \pi(y) - \pi(x) \sum_y \pi(y) f(y)\right)^2$

$$= \sum_{x,y} \pi(x)\pi(y)\left(f(x)^2 - f(x)f(y)\right) = \frac{1}{2}\sum_{x,y} \pi(x)\pi(y)\left(f(x)-f(y)\right)^2$$

  - $\mathcal{E}_\pi(f,f) \triangleq \frac{1}{2}\sum_{x,y} \pi(x)P(x,y)\left(f(x)-f(y)\right)^2$ (Dirichlet form or local variance)

$$= \langle \underbrace{(I-P)}_{\text{Laplacian}} f, f \rangle_\pi$$ (for reversible MC)

  - Laplacian $(I-P)$ has e-values $\beta_i = 1 - \lambda_i \Rightarrow 0 = \beta_1 < \beta_2 \le \dots \le \beta_n \le 2$

- **<u>Thm</u>**. For irreducible, reversible (P, π), we have

(Rayleigh's Thm in $\ell^2(\pi)$) $$\beta_2 = \inf\left\{ \frac{\mathcal{E}_\pi(f,f)}{\mathrm{Var}_\pi(f)} ;\ f \text{ non-constant} \right\}$$

<u>Note</u> - $\mathcal{E}_\pi(f + c\mathbb{1}, f + c\mathbb{1}) = \mathcal{E}_\pi(f,f)$. Thus, this is same as f s.t $\langle f, \mathbb{1}\rangle_\pi \neq 0$, ie, $\mathbb{E}_\pi[f] \neq 0$

Now we can use these to get geometric bounds

* Multicommodity Flow method (for general MC)

- Irreversible aperiodic ergodic MC (Ω, P), stat dist π
- For $e = (x,y)$, $C(e) = \pi(x) P(x,y)$ (capacity)

  For any $x,y \in \Omega^2$, $D(x,y) = \pi(x)\pi(y)$ (demand)
- Flow $f \equiv$ Routes $D(x,y)$ for all x, y
- $f : \mathcal{P} \to \mathbb{R}^+ \cup \{0\}$, $\mathcal{P} = \cup_{x,y} \mathcal{P}_{xy} \equiv \cup_{x,y}$ simple Paths from $x \to y$

$$\sum_{p \in \mathcal{P}_{x,y}} f(p) = D(x,y)$$

- $f(e) = \sum_{p : e \in p} f(p)$ - total flow on e, $\rho(f) = \max_e \frac{f(e)}{C(e)}$ - cost of f

  length of longest flow carrying path $\equiv \ell(f) = \max_{p : f(p) > 0} |p|$

Thm - For any lazy, ergodic MC P, flow f

$$t_{mix}(\varepsilon) \le O\left(\rho(f)\,\ell(f) \ln\left(1/\varepsilon\pi_{min}\right)\right)$$

- Any flow gives an upper bound
- Lower bounds can be derived from sparse cuts

* Conductance Bounds (for reversible MC)

- For Ergodic MC (Ω, P, π) define $\pi(A) = \sum_{x \in A} \pi(x)$ and ergodic flow $C(A) = \sum_{x \in A,\, y \notin A} \pi(x) P(x,y)$

(Note - $0 \leq C(A) \leq \pi(A) \leq 1$)

- The Conductance $\Phi(A) \triangleq \dfrac{C(A)}{\pi(A)}$

the conductance of P: $\Phi_*^* \triangleq \min_{A \mid \pi(A) \leq 1/2} \Phi(A)$

Thm (Jerrum-Sinclair '89) · For reversible MC

$$\frac{\Phi_*^2}{2} \leq 1 - \lambda_2 \leq 2\Phi_*$$

- This is sometimes referred to as Cheeger's Inequality
- Eg - LRW on hypercube - Consider $S = \{x \mid x^1 = 0\}$

$$\Rightarrow \Phi(S) = \frac{\sum_{x \in S, y \in S^c} 2^{-n} P(x,y)}{1/2} = 2^{-n+1} \cdot 2^{n-1} \cdot 1/2n = 1/2n$$

$\Rightarrow 2\Phi_* = 1/n = \gamma^* \Rightarrow$ Upper bound is tight

Eg - LRW on $2n$-cycle - $\Phi(S) = \dfrac{|\partial S|\,(1/4)(1/2n)}{|S|/2n} \leq \dfrac{1}{2n} = \Phi_*$ for $S = \{1, \dots, n\}$

$\Rightarrow \dfrac{\Phi_*^2}{2} = \dfrac{1}{8n^2}$ which is the correct order of γ^*