- **From last time**
  - Stoch matrix P has e-values $\lambda_1, \ldots, \lambda_n$ s.t $\lambda_1 = 1$ and $\underbrace{|\lambda_i| < 1}_{\text{if } P \text{ is ergodic}}$ $\forall i \geq 2$
  - If P is ergodic and reversible
    - λ_i are real, $1 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n > -1$

      $\gamma^* = 1 - \underbrace{\max\{|\lambda_2|, |\lambda_n|\}}_{\lambda^*}$
    - $$\frac{P^t(x,y)}{\pi(y)} = 1 + \sum_{j=2}^{n} \lambda_j^t v_j(x) v_j(y)$$ right e-vectors of P which are orthonormal in $\ell^2(\pi)$
    - $\left(\frac{1}{\gamma^*} - 1\right) \ln\left(\frac{1}{2\varepsilon}\right) \leq t_{mix}(\varepsilon) \leq \frac{1}{\gamma^*} \ln\left(\frac{1}{\pi_{min}\,\varepsilon}\right)$
    - If $\hat{P}$ is lazy (ie, $\hat{P} = \frac{1}{2}(I + P)$), then $\lambda^* = \lambda_2$, $\lambda_n \geq 0$

- We want to bound γ^* using 'network flows'
  - $\tau_{mix}(\varepsilon) = O\left(\rho(f)\,\ell(f) \ln\left(\frac{1}{\varepsilon \pi_{min}}\right)\right)$, where $\rho(f) = \max_e \frac{f(e)}{C(e)}$

    total flow on e, where f satisfies all demands $D(x,y) = \pi(x)\pi(y)$

    Ergodic flow $\pi(x)P(x,y)$

    Diaconis & Stroock '91, Sinclair '92 (For lazy version of general chains)

    $\ell(f)$ = length of longest flow-carrying path
  - For lazy reversible chains: $\frac{\Phi_*^2}{2} \leq \gamma^* \leq 2\Phi_*$ (Cheeger's Ineq)

    where $\underset{\text{Conductance}}{\Phi_*} = \min_{S \mid \pi(S) \leq 1/2} \left\{\frac{Q(S, S^c)}{\pi(S)}\right\}$, where $Q(S, S^c) = \sum_{\substack{x \in S \\ y \notin S}} \pi(x) P(x,y)$

• The Dirichlet Form

- For a fn f over Ω, its variance under π is

$$Var_\pi(f) = \sum_{x\in\Omega} \left(f(x) - \mathbb{E}_\pi f(x)\right)^2 \pi(x)$$
$$= \sum_{x\in\Omega} f(x)^2 \pi(x) - \left(\sum_x f(x)\pi(x)\right)^2$$
$$= \sum_x f(x)^2 \pi(x)\left(\sum_y \pi(y)\right) - \left(\sum_x f(x)\pi(x)\right)\left(\sum_y f(y)\pi(y)\right)$$
$$= \frac{1}{2} \sum_{x,y} \pi(x)\pi(y)\left(f(x)-f(y)\right)^2$$

- Similarly, the 'local variance' or Dirichlet form is given by

$$\mathcal{E}_\pi(f,f) = \frac{1}{2}\sum_{x,y} \pi(x)P(x,y)\left(f(x)-f(y)\right)^2 \quad \text{(for general MC)}$$
$$= \langle (I-P)f, f\rangle_\pi \quad \text{(for reversible Markov Chains)}$$

- Note - $\mathcal{E}_\pi(f,f) = \mathcal{E}_\pi(f+c, f+c)$ for any constant c

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Thm (Rayleigh's Characterization) - For ergodic, reversible MC (Ω, P, π),

$$1-\lambda_2 = \inf\left\{\frac{\mathcal{E}_\pi(f,f)}{Var_\pi(f)} ; f \text{ non-constant}\right\}$$

- Note - If f non-constant, then $Var_\pi(f) = \|f - \mathbb{E}_\pi f\|_2^2$ and $\mathcal{E}(f,f) = \mathcal{E}(f-\mathbb{E}f, f-\mathbb{E}f)$. Thus, the above is equivalent to $1-\lambda_2 = \inf\left\{\frac{\mathcal{E}_\pi(f,f)}{Var_\pi(f)} ; f\neq 0, \mathbb{E}_\pi f = 0\right\}$

- The proof follows from the same argument as the variational characterization of e-values via the Rayleigh quotient.

Upper bound of Cheeger's Inequality

$$\gamma^* = \inf_{f \mid \mathbb{E}_\pi f = 0} \frac{\sum_{x,y} \pi(x) P(x,y) \left(f(x)-f(y)\right)^2}{\sum_{x,y} \pi(x)\pi(y) \left(f(x)-f(y)\right)^2}$$

Now let $f_S = \begin{cases} -\pi(S^c) \ ; \forall x \in S \\ \pi(S) \ ; \forall x \notin S \end{cases}$. Check $\mathbb{E}_\pi[f] = 0$

S: subset of Ω s.t. $\pi(S) \leq 1/2$

$$\Rightarrow \gamma^* \leq \frac{\sum_{x\in S, y\notin S} \pi(x) P(x,y) \left(\pi(S^c)+\pi(S)\right)^2}{\sum_{x\in S, y \notin S} \pi(x)\pi(y) \left(\pi(S^c)+\pi(S)\right)^2}$$

$$= \frac{Q(S,S^c)}{\pi(S)\pi(S^c)} \leq \frac{2Q(S,S^c)}{\pi(S)} \quad (\because \pi(S^c) \geq 1/2)$$

Lower bound of Cheeger's Inequality

We first need an important lemma

Lemma ('The sweep algorithm') Given non-negative function $f : \Omega \to \mathbb{R}_+$, let $\Omega = \{x_{(1)}, x_{(2)}, \ldots, x_{(n)}\}$ be ordered in non-increasing order of f. Further, if $\pi[\{f > 0\}] \leq 1/2$, then $\mathbb{E}_\pi[f] \leq \Phi_*^{-1} \sum_{i<j} [f(x_{(i)}) - f(x_{(j)})] \cdot Q(x_{(i)}, x_{(j)})$

Pf – Let $S_t = \{x \in \Omega \mid f(x) > t\}$ for $t > 0$. Note $\pi(S_t) \leq 1/2$

$$\Rightarrow \Phi_* \leq \frac{Q(S_t, S_t^c)}{\pi(S_t)} = \frac{\sum_{x,y} Q(x,y) \mathbb{1}_{\{f(x) > t \geq f(y)\}}}{\pi(\{f > t\})}$$

Now we have $\mathbb{E}_\pi[f] = \int_0^\infty \pi(\{f > t\})\,dt$

$$\leq \Phi_*^{-1} \int_0^\infty \sum_{x,y} Q(x,y)\, \mathbb{1}_{\{f(x) > t \geq f(y)\}}\, dt$$

$$= \Phi_*^{-1} \sum_{x<y} Q(x,y)\, [f(x) - f(y)]$$

(Since $\int_0^\infty Q(x,y)\mathbb{1}_{\{f(x) > t \geq f(y)\}} dt = Q(x,y)(f(x)-f(y))$)

Pf of lower bound $\frac{\Phi_*^2}{2} \leq \gamma^*$

Let $f_2 \equiv$ right e-vector corresponding to λ_2. Assume $\pi(f_2 > 0) \leq \frac{1}{2}$ (else use $-f_2$)

Define $f = \max\{f_2, 0\} = f_2 + g$ ($f \geq 0$, $g \geq 0$)

Claim - $(I - P) f \leq \overbrace{(1-\lambda_2)}^{\gamma^*} f$ (i.e., $\forall x \in \Omega$, $[(I-P)f](x) \leq \gamma^* f(x)$)

To see this, consider two cases (Note $(I-P)f = \gamma^* f_2 + (I-P)g$)

i) $f(x) = 0$: Here $[(I-P)f](x) = [-Pf](x) \leq 0$ as $f \geq 0$

ii) $f(x) > 0$: Here $[(I-P)g](x) = [-Pg](x) \leq 0$ as $g \geq 0$

Thus, since $f \geq 0$, we have $\langle (I-P)f, f\rangle_\pi \leq \gamma^* \langle f, f\rangle_\pi$

$\Rightarrow \gamma \geq \dfrac{\langle (I-P)f, f\rangle_\pi}{\langle f, f\rangle_\pi}$. Now from the previous lemma (with f^2)

we have $\langle f, f\rangle_\pi^2 \leq \Phi_*^{-2} \left[\sum_{x<y} [f^2(x) - f^2(y)]\, Q(x,y)\right]^2$

(C-S Ineq) $$\leq \Phi_*^{-2} \left[\sum_{x<y} [f(x) - f(y)]^2 Q(x,y)\right]\left[\sum_{x<y} [f(x)+f(y)]^2 Q(x,y)\right]$$

$$\leq \Phi_*^{-2}\, \mathcal{E}_\pi(f,f)\left[2\langle f,f\rangle_\pi - \mathcal{E}_\pi(f,f)\right]$$

Let $R = \mathcal{E}_\pi(f,f)/\langle f,f\rangle_\pi$. Dividing above by $\langle f,f\rangle_\pi^2$, we get

$$\Phi_*^2 \le R(2-R) \Rightarrow 1-\Phi_*^2 \geq (1-R)^2 \geq (1-\gamma)^2$$

Also $\left(1-\frac{\Phi_*^2}{2}\right)^2 \geq 1-\Phi_*^2 \Rightarrow \frac{\Phi_*^2}{2} \le \gamma^*$ $\square$

Now we see how to extend to general (lazy) MC

<u>Lemma</u> - For ergodic MC P, and lazy variant $(\hat{P} = \frac{1}{2}(I+P), \pi)$ for any $f:\Omega \to \mathbb{R}$, we have $Var_\pi[\hat{P}f] \le Var_\pi[f] - \mathcal{E}_\pi(f,f)$

<u>Note</u> - Let $\gamma^* \triangleq \inf_{f:\text{non-constant}} \frac{\mathcal{E}_\pi(f,f)}{Var_\pi[f]}$. Then the above result implies that $Var_\pi[\hat{P}^t f] \le (1-\gamma^*)^t Var_\pi[f] \le e^{-\gamma^* t} Var_\pi[f]$

Now let $f = \mathbb{1}_A$ and suppose we start at x_0

- $Var_\pi[\hat{P}^t f] \le e^{-\gamma^* t}\pi(A)(1-\pi(A)) \le e^{-\gamma^* t}/4 \le \varepsilon^2 \pi(x_0)$

  for $t = \frac{1}{\gamma^*}\left(\ln\left(\frac{4}{\pi(x_0)}\right) + 2\ln\left(\frac{1}{\varepsilon}\right)\right)$

- $Var_\pi[\hat{P}^t f] \geq \pi(x_0)\Big[\underbrace{[\hat{P}^t f](x_0)}_{e_x P^t(A)} - \underbrace{E_\pi[P^t f]}_{\pi(A)}\Big]^2 = \pi(x_0)\left(P^t(x_0,A) - \pi(A)\right)^2$

$\Rightarrow \underbrace{\forall A \subseteq \Omega, \quad |P^t(x,A) - \pi(A)| \le \varepsilon}_{\text{i.e., } d(t) \le \varepsilon}$ for $t \geq \frac{1}{\gamma^*}\left(\ln\left(\frac{4}{\pi(x_0)}\right) + 2\ln\left(\frac{1}{\varepsilon}\right)\right)$

- Pf of Lemma

- $$\left[\hat{P} f\right](x) = \frac{f(x)}{2} + \frac{1}{2}\sum_y P(x,y) f(y) = \frac{1}{2}\sum_y P(x,y)\big(f(x)+f(y)\big)$$

- WLOG, assume $\mathbb{E}_\pi[f] = 0$ (constant shifts don't affect $\mathcal{E}_\pi$, Var_π)

$$\Rightarrow \mathrm{Var}_\pi(Pf) = \sum_x \pi(x)\Big(\underbrace{\tfrac{1}{2}\sum_y P(x,y)\big(f(x)+f(y)\big)\Big)^2}_{\wedge\!\backslash\ \text{by Jensen's}}$$

$$\leq \frac{1}{4}\sum_{x,y} \pi(x) P(x,y)\big(f(x)+f(y)\big)^2$$

$\because \hat{P} = \frac{1}{2}(I+P)$, $\pi\hat{P} = \pi$

$\Rightarrow \frac{\pi(y)}{2} = \frac{1}{2}\sum_x \pi(x) P(x,y)$

Also $$\mathrm{Var}_\pi(f) = \frac{1}{2}\sum_x \pi(x) f(x)^2 + \frac{1}{2}\sum_y \pi(y) f(y)^2$$

$$= \frac{1}{2}\sum_{x,y} \pi(x) P(x,y)\big(f(x)^2 + f(y)^2\big)$$

$$\Rightarrow \mathrm{Var}_\pi[f] - \mathrm{Var}_\pi[\hat{P}f] \geq \frac{1}{4}\sum_{x,y} \pi(x) P(x,y)\big(f(x)-f(y)\big)^2 = \mathcal{E}_\pi(f,f)$$

Note - The $\frac{I+P}{2}$ form ensures periodicity; a similar trick can be done by embedding the chain P in a 'faster' continuous time chain

- Can also show $\tau_{mix}(\varepsilon) \geq \left(\frac{1}{\gamma^*} - 1\right)\ln\left(\frac{1}{2\varepsilon}\right)$ as before

- Now we want to bound γ^* in terms of flows

Thm -
$$\inf_{g \text{ non-constant}} \frac{\mathcal{E}_\pi(g,g)}{\mathrm{Var}_\pi(g)} \geq \frac{1}{\rho(f)\,\ell(f)}$$

<u>Pf</u> - $\mathrm{Var}_\pi(g) = \frac{1}{2} \sum_{x,y} \underbrace{\pi(x)\pi(y)}_{D(x,y)} (g(x)-g(y))^2$

$$= \frac{1}{2} \sum_{x,y} \underbrace{\sum_{p \in P_{xy}} f(p)}_{\text{flow satisfying } D(x,y)} (g(x)-g(y))^2$$

- For any path $p \in P_{xy}$, $g(x)-g(y) = \sum_{(u,v)\in p} (g(v)-g(u))$

$$\Rightarrow 2\mathrm{Var}_\pi(g) = \sum_{x,y} \sum_{p\in P_{xy}} f(p) \Big(\sum_{(u,v)\in p} g(v)-g(u) \Big)^2$$

$$\leq \sum_{x,y} \sum_{p \in P_{xy}} f(p)\,|p| \Big(\sum_{(u,v)\in p} (g(v)-g(u))^2 \Big) \quad \text{(CS Ineq)}$$

$$= \sum_{e=(u,v)} (g(v)-g(u))^2 \overbrace{\sum_{p \ni e} f(p)\,|p|}^{f(e)\,;\ \rho(f) = \max_e \frac{f(e)}{C(e)}\text{, where } C(e)=\pi(u)P(u,v)}$$

$$\leq \ell(f)\rho(f) \sum_{e\in(u,v)} (g(v)-g(u))^2\, C(e)$$

$$= 2\,\ell(f)\,\rho(f)\ \mathcal{E}_\pi(g,g)$$