Mixing Times via Canonical Flows

- **Setup** – Demands $D(x,y) = \pi(x)\pi(y) \quad \forall x,y \in \Omega$

  Capacities $C(x,y) = \pi(x) P(x,y)$

  Flow $f : \bigcup_{x,y} P_{x,y} \longrightarrow \mathbb{R}_+ \cup \{0\}$ s.t. $\sum_{p \in P_{x,y}} f(p) = D(x,y)$

  paths from $x \to y$

  Cost $\rho(f) = \max_e \frac{f(e)}{C(e)}$, $\sum_{e \in p} f(p)$, length $\ell(f)$ = longest flow-carrying path

- **Poincaré Constant** – $\gamma^* = \inf_{f:\text{non constant}} \left[\frac{\mathcal{E}_\pi(f)}{\mathrm{Var}_\pi(f)} \right]$

- **Flow bound** – $t_{mix}(\varepsilon) \leq \frac{1}{\gamma^*} \left[2\ln(1/\varepsilon) + \ln(1/\pi(x_0)) \right]$

  $\gamma^* \geq \frac{1}{\rho(f)\,\ell(f)}$ (for lazy, ergodic MC)

Converses – $t_{mix}(\varepsilon) \geq \left(\frac{1}{\gamma^*} - 1\right) \ln\left(\frac{1}{2\varepsilon}\right)$, $\gamma^* \leq \text{constant} \cdot \frac{\log n}{\rho^U(f)}$

(Leighton-Rao '88)

For reversible MC

$$\frac{\Phi_*^2}{2} \leq \gamma^* \leq 2\Phi_*, \text{ where } \Phi_* = \inf_{S \mid \pi(S) \leq 1/2} \frac{Q(S,S^c)}{\pi(S)}$$

<u>Eg</u> - LRW on hypercube. Let $N = |\Omega| = 2^n$, $n \equiv$ dim of hypercube

- $C(u,v) = \frac{1}{N} \cdot \frac{1}{2n}$, $D(x,y) = \frac{1}{N^2}$

- Consider $D(x,y)$ <u>equally split</u> among all shortest paths
  - By symmetry, $f(e)$ is equal $\forall e = (u,v)$

$$\begin{tikzcd}[column sep=small, row sep=small] & 110 \arrow[dr] & \\ 010 \arrow[ur] \arrow[dr] & & 111 \\ & 011 \arrow[ur] & \end{tikzcd}$$

  - Total flow $\sum_e f(e) = \frac{1}{N^2} \sum_{x,y} |\text{shortest path } x \to y|$

$$= \frac{1}{N^2} \cdot \left(N^2 \cdot \frac{n}{2}\right) = \frac{n}{2} \Rightarrow f(e) = \frac{n}{2} \cdot \frac{1}{Nn}$$

$|E| = Nn$, directed paths

  - $\rho(f) = \frac{1/2N}{1/2Nn} = n$, $\ell(f) = n \Rightarrow t_{mix}(\varepsilon) \leq n^2 (\ln N + \ln 1/\varepsilon) = O(n^3)$

- <u>Note</u> - We know $t_{mix}(\varepsilon) = O(n \log n)$!
  - However $1 - \lambda_2 = 1/n \Rightarrow$ e-value bounds in general give $t_{mix} = O(n^2)$!

    Loss due to ignoring higher order terms (i.e., $\lambda_3, \lambda_4, \ldots$)
  - Moreover, the best flow bound gives $\gamma^* \geq 1/n^2$

<u>Eg</u> - LRW on line. $|\Omega| = N = n$, $C(u,v) = \frac{1}{4N}$, $D(x,y) = \frac{1}{N^2}$

- Let f be flow on unique path $\Rightarrow f(i,i+1) = i \cdot (N-i) \cdot \frac{1}{N^2} \leq \frac{1}{4}$

$\Rightarrow \rho(f) \leq N$, $\ell(f) = N \Rightarrow t_{mix} \leq N^2 (\ln(\frac{1}{\varepsilon}) + \ln N) = O(N^2 \ln N)$

- Here, true $t_{mix} = O(N^2) \Rightarrow$ off only by $\ln N$

General Schema for $O(\text{poly}(n))$ bounds

- Suppose $|\Omega| = N = O(e^n)$, $n \equiv$ "natural dim^n" of P
- $P(x,y) \geq \frac{1}{\text{poly}(n)}$, $\pi(x) = \frac{1}{N}$ $\Rightarrow$ $C(u,v) = \frac{1}{N\text{poly}(n)}$, $D(x,y) = \frac{1}{N^2}$
- $\ell(f) \leq \text{poly}(n)$, $\rho(f) \leq \text{poly}(n)$ $\Rightarrow t_{mix} = O(\text{poly}(n))$

$\Rightarrow$ we need $\boxed{f(e) \leq \frac{\text{poly}(n)}{N}}$

- Now since $|E| \approx N.\text{poly}(n)$, $\sum f(e) \simeq 1 \Rightarrow \exists e$ s.t $f(e) \geq \frac{1}{N\text{poly}(n)}$

$\Rightarrow$ optimizing f can not give better than poly(n)!

- Suppose we send $D(x,y)$ along a <u>single path</u> β_{xy}
  - Let $P_e \equiv \{(x,y) \mid \beta_{xy} \text{ contain } e\}$

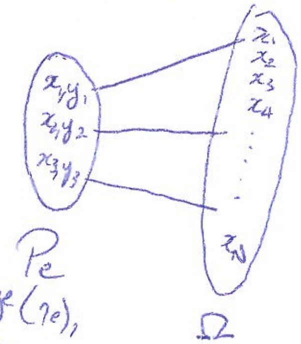
    $\Rightarrow f(e) = \frac{|P_e|}{N^2}$ $\Rightarrow$ <u>Need $|P_e| \leq \text{poly}(n).N \ \forall e$</u>

- Problem - May not know N. Eg hard-core model

- Solution - Construct injective map $\eta_e : P_e \to \Omega$
  - Separate fn η_e for each $e = (u,v)$
  - Injective $\Rightarrow$ Can invert $\eta_e(x,y)$, ie,

    given $\omega \in \Omega$, and $e=(u,v)$, if $x \in$ range (η_e),

    can find unique (x,y) s.t $\eta_e((x,y)) = \omega$



Eg - LRW on hypercube

- Let $\beta_{xy} \equiv$ left-to-right bit fixing path

- Let $e = (u, v)$ be s.t u and v differ in i^{th} position

$$\underset{x}{0110} \to \overset{u}{1110} \xrightarrow{e} \overset{v}{1010} \to \underset{y}{1011}$$

  - suppose $x = (x_1 x_2 \dots x_n)$, $y = (y_1, y_2, \dots, y_n)$ be s.t $e \in \beta_{xy}$

  - set $\eta_e((x,y)) = x_1 x_2 \dots x_i \, y_{i+1} y_{i+2} \dots y_n$. Given e and $\omega \in \Omega$, clearly can recover $\eta_e^{-1}(\omega)$ if $\omega \in \text{Range}(\eta_e)$

- Thus $|P_e| \leq N \Rightarrow \rho(f) \leq \dfrac{|P_e|/N^2}{1/2nN} = 2n$

$$\ell(f) \leq n$$

Note - Same as even split f up to constants

- Defn - A flow encoding for flow f, that uses single paths $\beta_{xy}\ \forall x, y$, is a set of fns $\eta_e : P_e \to \Omega$, one for each edge $e = (u, v)$ s.t

i) η_e is injective, ii) $\pi(x)\pi(y) \leq \beta \pi(u) \pi(\eta_e(x,y))$ $\forall (x,y) \in P_e$ for $e = (u,v)$

(or approx injection}, ie, can store extra info to invert)

- Prop^n - If $\exists$ flow encoding $\eta_e \Rightarrow \rho(f) \leq \beta \max\limits_{(u,v) = e \mid P(u,v) > 0} \left[\dfrac{1}{P(u,v)}\right]$

Pf - For any $e = (u,v)$, $f(e) = \sum\limits_{(x,y) \in P_e} \pi(x)\pi(y) \underset{\text{Prop 2}}{\leq} \beta \pi(u) \sum\limits_{x,y \in P_e} \pi(\eta_e(x,y)) \underset{\text{Prop 1}}{\leq} \beta \pi(u)$

$\Rightarrow \rho(f) = \max \dfrac{f(e)}{c(e)} \leq \beta \max\limits_{e} \left(\dfrac{1}{P(u,v)}\right)$

* Sampling random matchings of G

- Directed graph $G(V,E)$, $\lambda > 1$

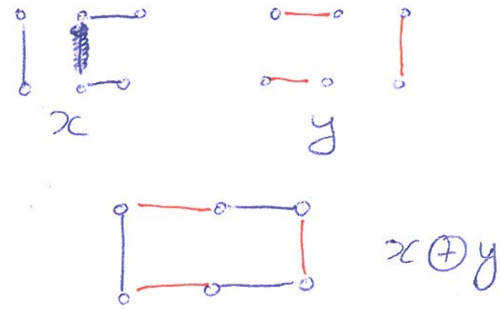
  Want to sample matching M wp $\pi(M) = \frac{\lambda^{|M|}}{Z}$ (Gibbs Measure)

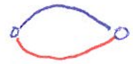
- Suppose m_k = # of k matchings of G $\Rightarrow$ $Z = \sum_{k=0}^{\infty} \lambda^k m_k$
  - Computing Z is #-P complete for any $\lambda > 0$!

- Markov Chain - Lazy MH sampler
  - i) wp $1/2$, no change
  - ii) Else, choose edge $e = (u,v)$ u.a.r
    - Add if possible (ie, if $M+e$ is a matching)
    - If $e \in M$, discard e (ie $M \to M-e$) wp $1/\lambda$ — Metropolis Rule!
    - If exactly one of u or v is matched in M, (say by edge e'), then Swap (ie., $M \to M-e'+e$)

- Check - the above chain uniformly samples neighboring matchings of M (ie, which differ in at most 2 edges), transitions via the Metropolis rule (Note - $\lambda > 1$)

Flow - Let $x, y \equiv$ matchings. $x \oplus y \equiv$ Superposition



- $x \oplus y$ comprises of
  - Double edges 
  - (Even) alternating ~~congruent~~ cycles
  - Alternating paths 

- For analysis - Fix a total ordering on all simple paths and even cycles subgraphs of G, and designate a 'start' vertex in each subgraph (endpoint for path, any node on cycle + direction)

- To find $P_{xy} \equiv$ Construct $x \oplus y$, and 'fix' each alternating path and cycle in order given by our ordering.

Eg -